How to Intonate a Guitar

Contents

Prerequisites

None.

Notes

None.

Document History

Date	Version	Comments
15th July 2014	1.0	Initial creation of the document.
24th July 2014	1.1	Cosmetic changes.

1 Introduction

The traditional method for intonating guitars is to adjust the bridge position for each string until the notes played on the open strings and at the 12^{th} frets are perfectly in tune.

It is assumed that any errors induced by this process are evenly spread across all the other fretted notes. This, of course, is very unlikely to be the case.

Apart from altering fret positions to get notes perfectly tuned[1], the only other variable that is relatively easy to change is the position of the nut for each string. See, for example, Hartley (Current), and his "Funky Nut".

This paper develops a simple mathematical model for a guitar string, and using this model, determines where the ideal nut and bridge position should be for any given string and player characteristics, so that the errors in frequencies of notes played at all frets are minimised.

2 Mathematical Model of a Guitar String

The model follows the lead of Lane and Kasparis (2012) and Varieschi and Gower (2010).

2.1 String Tension

We need to understand what factors affect string tension, and how string tension affects the things we are interested in, notably the frequency of the note produced by the string.

Theory says that the solution of the wave equation that applies to waves on strings,

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho} \frac{\partial y}{\partial x^2}$$

(where y is the displacement of the string at a point x along its length when the tension in the string is T and the mass per unit length of the string is ρ) is given by

$$T = (2fS)^2 \rho \tag{1}$$

where S is the length of the vibrating string and f is the fundamental frequency of vibration of the string.

This formula shows the link between the length of the vibrating string, the frequency of the note played by the string, and the string tension. It's a really important equation.

Usually, the tension in the string is assumed to be constant. But of course, there's Hooke's Law...

2.2 Hooke's Law

Hooke's Law tells us that if we have a guitar string, and we hold it down against a fret, then we will be increasing the tension in the string slightly. The reason for that is because when we fret a string we are in fact extending the string a bit. And Hooke's Law tells us the relationship between the extension and the tension that we add:

$$\Delta T = EA\frac{\Delta l}{l} \tag{2}$$

where ΔT is the small increase in tension that we add to the string (of Young's Modulus E and cross-sectional area A) when we extend a string of length l by the small amount Δl.

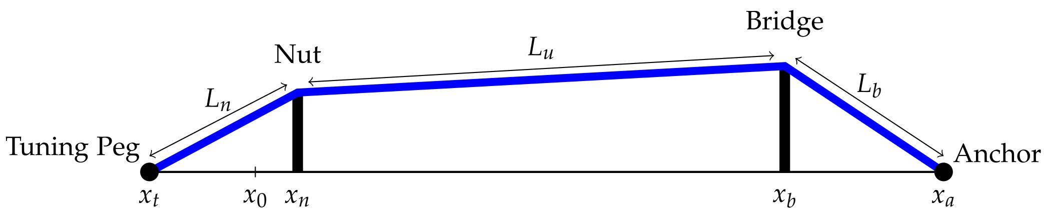
That means that when you tune your unfretted string to a perfect note (where the string tension will be T), then hold down a fret, the tension in the string is now $T + \Delta T$, so with a perfectly manufactured guitar, where the nut, bridge and frets are perfectly positioned, the note you play will be *sharp*. Bugger.

[1]Something that has been done. See, for example, True Temperament (2014). This is not an ideal solution for most guitarists, however. Necks produced with equally tempered frets are costly and do not easily lend themselves to some techniques used by guitarists, such as string bending.

2.3 The Model

In order to try and understand how we can get around this problem, I'm going to produce a mathematical model of a guitar string. It's a relatively simple model, but I'm hoping it's going to give me the results I need to help me design and intonate guitars. So here goes.

Figure 1: My Model of a Guitar String : The Unfretted String



The basic features of my model are shown in Figures 1 and 2. The string is held down at fixed points by a tuning peg at one end, and an anchor at the other. It passes over a nut and a bridge, which allow the string to slide over them. The positions of the tuning peg, origin, nut, bridge and anchor are x_t, x_0, x_n, x_b and x_a respectively.

In this model, the variables that the guitarist can control are: the nut height and nut position, bridge height and bridge position, and the tension in the string. So as well as being able to adjust the "saddle setback", i.e. the bridge position, the guitarist can also adjust the position of the nut. This is why I am not assuming that the position of the nut is at x_0.

Figure 2: My Model of a Guitar String : The Fretted String

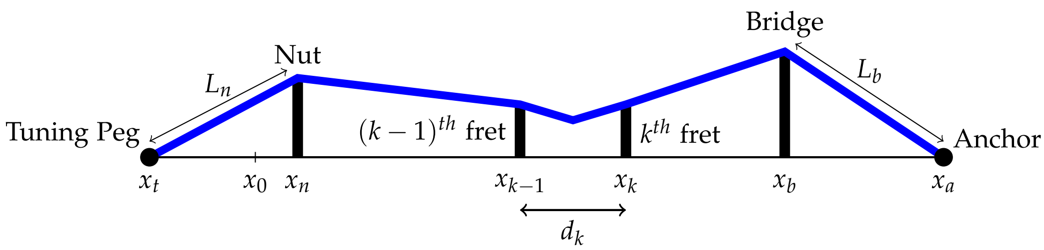


Figure 2 shows the string when it is fretted at the k^{th} fret. I'll go into more detail about how this is done shortly, but the main point to be aware of here is that the string is pinned down between the fret being played (the k^{th} fret), and the previous one (the $(k-1)^{th}$ fret). I am assuming that all frets are positioned along the x-axis according to traditional positioning (see, for example, Smith (2014)).

That means that the k^{th} fret is at position

$$x_k = S\left(1 - \frac{1}{r^k}\right) \tag{3}$$

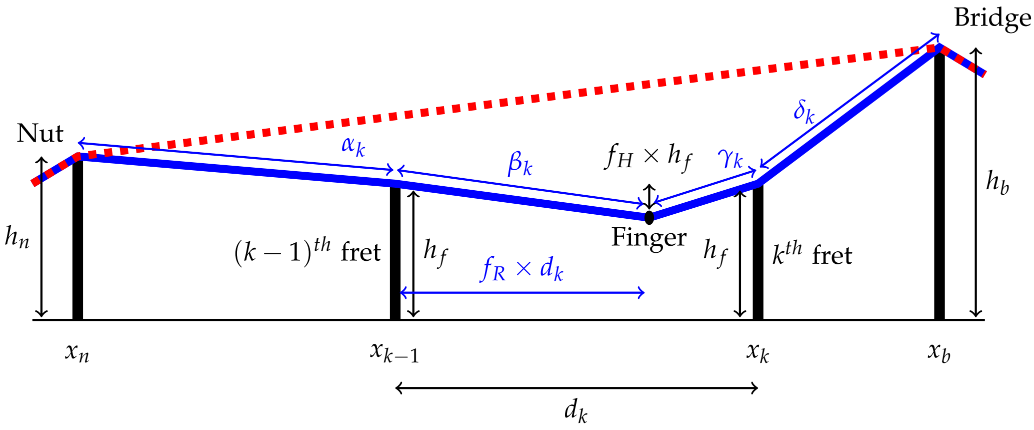
where S is the required scale length for the instrument, and $r = 2^{\frac{1}{12}}$. I've also called d_k the distance between the $(k-1)^{th}$ fret and the k^{th} fret.

2.4 Fretting a String : The General Case

The model for fretting a string is shown in Figure 3. When fretting most frets, the finger holds down the string so that it pins the string between two frets: the fret used to provide the note to be played, and the next one toward the nut.

In my model there are adjustable factors, f_H and f_R that can be used to change the way a string is fretted. f_H represents the fraction of the total fret height that the finger pulls the string down below the height of

Figure 3: Fretting a String



a fret; f_R represents the fraction of the fret spacing where the finger is located when fretting the string. These factors can be used to model the player's style: you can increase f_H, for example, if a player has a heavy touch; reduce f_H for a light touch. In the first version of this model, f_H and f_R are modelled as constants. But they can be modelled as functions of other factors, of course, such as fret number.

When the string is not fretted, I am assuming in this model that it is held perfectly straight by the tension in the string between the top of the nut and the top of the bridge (Figure 1). In that case the total length of the string, L_{total}, will be

$$\begin{aligned} L_{total} &= L_n + L_u + L_b \\ &= L_n + \left\{(x_b - x_n)^2 + (h_b - h_n)^2\right\}^{\frac{1}{2}} + L_b \end{aligned}$$

where L_n is the distance from the tuning peg to the top of the nut, and L_b is the distance from the top of the bridge to the anchor. L_u is the unfretted length of the string between the top of the nut and the top of the bridge. h_n is the height of the nut, and h_b is the height of the bridge.

When fretted, the string is extended, by the action of the player's finger, into the straight lines shown in Figure 3. Its length, $L_{f,k}$, when fretted at position k will now be

$$L_{f,k} = L_n + \alpha_k + \beta_k + \gamma_k + \delta_k + L_b$$

where, by simple geometry and extensive use of Pythagoras' Theorem,

$$\begin{aligned} \alpha_k &= \left\{(x_{k-1} - x_n)^2 + (h_n - h_f)^2\right\}^{\frac{1}{2}} \\ \beta_k &= \left\{(f_R d_k)^2 + (f_H h_f)^2\right\}^{\frac{1}{2}} \\ \gamma_k &= \left\{([1 - f_R] d_k)^2 + (f_H h_f)^2\right\}^{\frac{1}{2}} \\ \delta_k &= \left\{(x_b - x_k)^2 + (h_b - h_f)^2\right\}^{\frac{1}{2}} \end{aligned}$$

Consequently, when the string is fretted at fret k, it is extended by an amount

$$E_k = L_{f,k} - L_u = \alpha_k + \beta_k + \gamma_k + \delta_k - L_u \tag{4}$$

And so, using Hooke's Law (2), the tension in the string when fretted at position k will be

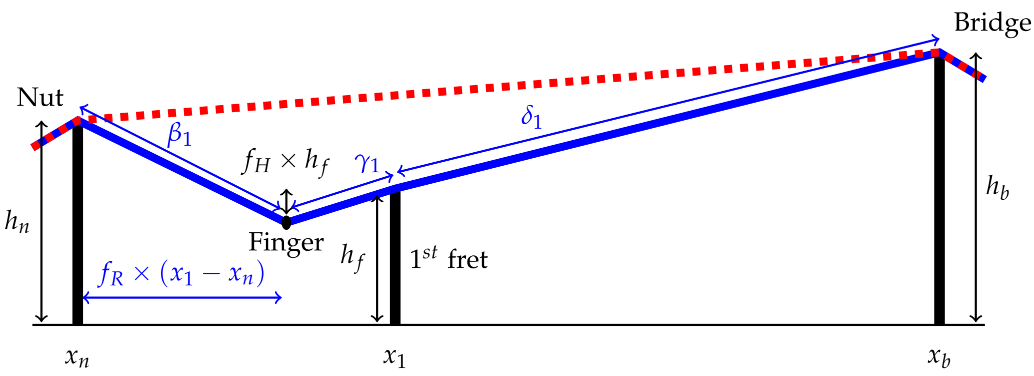
$$T_k = T + EA\frac{E_k}{L_u} \tag{5}$$

which means that the note played at fret k will now be sharp, unless the tension in the string is reduced, but then the unfretted note of the open string will be flat...

2.5 Fretting a String : Near the Nut

There are two additional features that I would like to build into my model. The first is that the string extension calculated in Section 2.4 wouldn't apply for the first fret. At the first fret, the *previous* fret will be the *nut*, so we have to adapt the string extension calculation. Figure 4 shows you what I mean. So at

Figure 4: Fretting a String at the First Fret



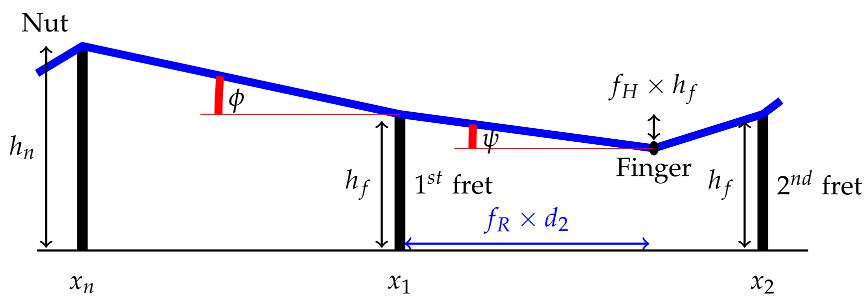
the first fret,

$$
\begin{aligned}
\alpha_1 &= 0 \\
\beta_1 &= \left\{(f_R(x_1 - x_n))^2 + (h_n - [1 - f_H]h_f)^2\right\}^{\frac{1}{2}} \\
\gamma_1 &= \left\{([1 - f_R](x_1 - x_n))^2 + (f_H h_f)^2\right\}^{\frac{1}{2}} \\
\delta_1 &= \left\{(x_b - x_1)^2 + (h_b - h_f)^2\right\}^{\frac{1}{2}}
\end{aligned}
$$

Now, could there be a similar problem at the second fret (or third...)?

At the second fret, we can't use the string extension calculated in Section 2.4 if the angle of the string from the first fret to the nut (ϕ) is greater than the angle of the string from the finger to the first fret (ψ) (see Figure 5).

Figure 5: Fretting a String at the Second Fret



In other words, what we need to do is to compare $\tan \phi$ with $\tan \psi$. If we do this for the situation where the finger is between the $(k-1)^{th}$ fret and the k^{th} fret, then

$$\tan(\phi) = \frac{h_n - h_f}{x_{k-1} - x_n}$$

and

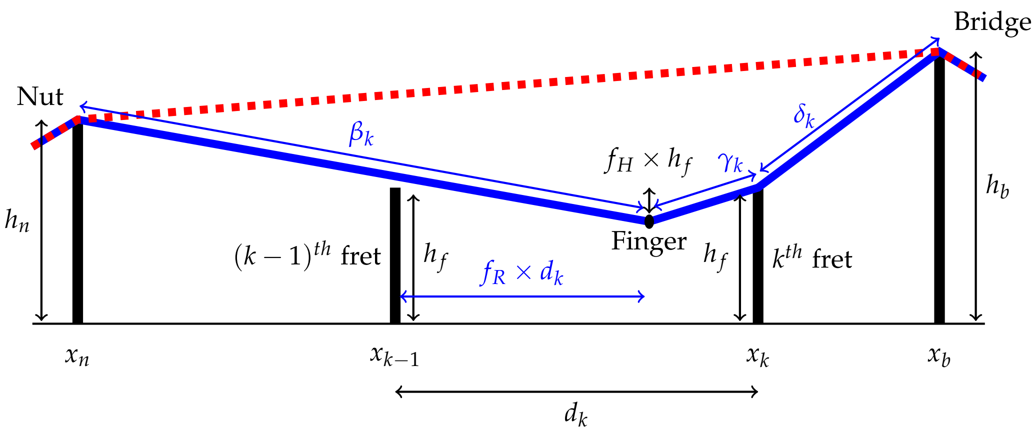
$$\tan(\psi) = \frac{f_H h_f}{f_R d_k}$$

So if we calculate the value

$$G_k = \frac{\tan(\phi)}{\tan(\psi)} = \frac{h_n - h_f}{x_{k-1} - x_n} \times \frac{f_R d_k}{f_H h_f} \tag{6}$$

then we can compare this with 1. If $G_k > 1$, then we must use a different method to calculate the extension of the string.

Figure 6: Fretting a String Near the Nut



This time the lengths of the extended string will be given by:

$$\begin{aligned} \alpha_k &= 0 \\ \beta_k &= \left\{(x_{k-1} - x_n + f_R d_k)^2 + (h_n - [1 - f_H]h_f)^2\right\}^{\frac{1}{2}} \\ \gamma_k &= \left\{([1 - f_R]d_k)^2 + (f_H h_f)^2\right\}^{\frac{1}{2}} \\ \delta_k &= \left\{(x_b - x_k)^2 + (h_b - h_f)^2\right\}^{\frac{1}{2}} \end{aligned}$$

Putting this all together we get the string length equations to be:

$$\alpha_k = \begin{cases} 0 & : \text{ near the nut} \\ \left\{(x_{k-1} - x_n)^2 + (h_n - h_f)^2\right\}^{\frac{1}{2}} & : \text{ away from the nut} \end{cases} \tag{7}$$

$$\beta_k = \begin{cases} \left\{(f_R(x_1 - x_n))^2 + (h_n - [1 - f_H]h_f)^2\right\}^{\frac{1}{2}} & : \text{ at the first fret} \\ \left\{(x_{k-1} - x_n + f_R d_k)^2 + (h_n - [1 - f_H]h_f)^2\right\}^{\frac{1}{2}} & : \text{ near the nut} \\ \left\{(f_R d_k)^2 + (f_H h_f)^2\right\}^{\frac{1}{2}} & : \text{ away from the nut} \end{cases} \tag{8}$$

$$\gamma_k = \begin{cases} \left\{([1 - f_R](x_1 - x_n))^2 + (f_H h_f)^2\right\}^{\frac{1}{2}} & : \text{ at the first fret} \\ \left\{([1 - f_R]d_k)^2 + (f_H h_f)^2\right\}^{\frac{1}{2}} & : \text{ everywhere else} \end{cases} \tag{9}$$

$$\delta_k = \left\{(x_b - x_k)^2 + (h_b - h_f)^2\right\}^{\frac{1}{2}} \quad : \text{ everywhere} \tag{10}$$

where "near the nut" is determined by equation (6).

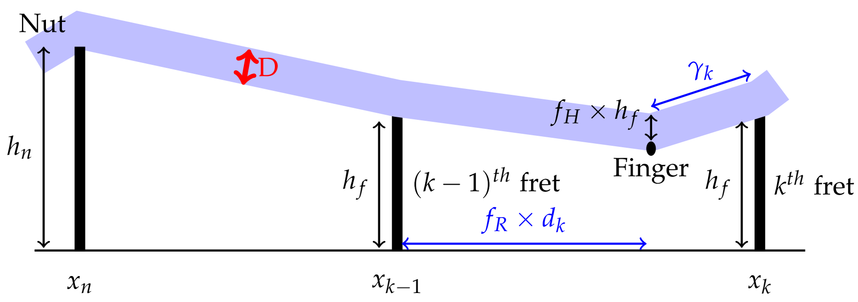
2.6 Fretting a String : Final Thoughts

The final bit of the model that so far I have not considered is the dimensions of the string. Some strings are thicker than others. Some strings have a core with a winding around them. How can we model this?

Well, there are three issues with string thickness. The first is that string thickness is important when we want to calculate the cross-sectional area that's used in the Hooke's Law calculation. For this, we are only interested in the part of the string that resists when you stretch it. That will be the *core* of the string if the string is wound, and the whole string if the string is not wound.

The second issue is the effect of the thickness of the string in calculating the extension of the string. Using

Figure 7: Thinking About String Thickness



the same kind of picture (Figure 7) that we have been using already when working out string extensions, then if we want to work out the length γ_k, then it will be:

$$\gamma_k = \left\{ ([1 - f_R]d_k)^2 + \left(\left[h_f + \frac{D}{2} \right] - \left[[1 - f_H]h_f + \frac{D}{2} \right] \right)^2 \right\}$$

since the height of the core of the string above the k^{th} fret will be $h_f + \frac{D}{2}$ because the string is almost horizontal. This simplifies to

$$\gamma_k = \left\{ ([1 - f_R]d_k)^2 + \left(f_H h_f \right)^2 \right\}$$

and the tickness of the string has cancelled itself out.

The third issue with string thickness is that the whole of the string, and not just the core, will affect the mass of the string per unit length, ρ.

2.7 Using the New Tension

Equations (4), (7), (8), (9) and (10) can be used to find the extension in the string when the string is fretted at any fret.

Then, using Equation (5), it is possible to work out the new tension in the string. Using this new tension, you can work out the frequencies of the notes that will be played on the open string, and all the frets.

Let's say that F_k is the frequency required to be produced at the k^{th} fret. And let's say that f_k is the frequency actually produced at the k^{th} fret. Having calculated the produced frequencies, we can work out the error in each frequency. Small differences in frequencies are usually quoted in *cents* (see, for example, Wikipedia (Current)), the definition of which is

$$\Delta f = 1200 \log_2 \left(\frac{f_1}{f_2} \right) \tag{11}$$

If one frequency is double the other, then the number of cents between the two frequencies will be 1200. For the Western scale, with 12 notes in an octave, and the note one octave up will have double the frequency, then a cent will be one hundredth of the interval between one note and the next.

So the error, measured in cents, between the actual and the required frequencies at each fret will be

$$\Delta F_k = 1200 \log_2 \left(\frac{f_k}{F_k} \right) \tag{12}$$

ΔF_k will be negative if the note is flat, and positive if the note is sharp.

The name of the game then, is to adjust T, the tension in the unfretted string, x_b, the bridge position, and x_n, the nut position, so that when you play notes at all the frets, the errors are minimised.

3 Intonation Algorithm

3.1 Determine Necessary Constants

The following constants are used within the algorithm:

- S, the required scale length of the guitar;
- $r = 2^{\frac{1}{12}}$;
- h_n, the height of the nut above the fretboard;
- h_f, the height of the top of the frets above the fretboard;
- h_b, the height of the bridge above the fretboard;
- f_H, the fraction of the height of the frets that the player pulls the string down below the height of the frets when fretting a string;
- f_R, the fraction of the distance between frets where the player frets a string;
- $L_u = \left\{(x_b - x_n)^2 + (h_b - h_n)^2\right\}^{\frac{1}{2}}$, the unfretted distance from the top of the nut to the top of the bridge;
- L_n, the length of the string between the tuning peg and the nut;
- L_b, the length of the string between the bridge and the anchor;
- $L_{total} = L_n + L_u + L_b$, the length of the string between the tuning peg and the anchor when unfretted;
- F_k, the frequencies of the notes that are required at each fret;
- f_t, the frequency error threshold, in cents, for the player. Get the frequency error at all frets below f_t, and the guitar is in tune!

3.2 Calculate Fret Positions

Calculate the position of the k^{th} fret using (3):

$$x_k = S\left(1 - \frac{1}{r^k}\right)$$

where S is the scale length chosen for the guitar, and $r = 2^{\frac{1}{12}}$.

3.3 Calculate the Spaces Between Frets

Calculate the spaces between the frets using:

$$d_k = x_k - x_{k-1}$$

3.4 Calculate the "Near Nut Factor"

Calculate the "near nut factor", G_k, using (6)

$$G_k = \frac{h_n - h_f}{x_{k-1} - x_n} \times \frac{f_R d_k}{f_H h_f}$$

3.5 Calculate the String Extensions

Calculate the string extension lengths using (7), (8), (9) and (10):

$$\alpha_k = \begin{cases} 0 & : \ G_k > 1 \\ \left\{(x_{k-1} - x_n)^2 + (h_n - h_f)^2\right\}^{\frac{1}{2}} & : \ G_k < 1 \end{cases}$$

$$\beta_k = \begin{cases} \left\{(f_R(x_1 - x_n))^2 + (h_n - [1 - f_H]h_f)^2\right\}^{\frac{1}{2}} & : \ k = 1 \\ \left\{(x_{k-1} - x_n + f_R d_k)^2 + (h_n - [1 - f_H]h_f)^2\right\}^{\frac{1}{2}} & : \ G_k > 1 \\ \left\{(f_R d_k)^2 + (f_H h_f)^2\right\}^{\frac{1}{2}} & : \ G_k < 1 \end{cases}$$

$$\gamma_k = \begin{cases} \left\{([1 - f_R](x_1 - x_n))^2 + (f_H h_f)^2\right\}^{\frac{1}{2}} & : \ k = 1 \\ \left\{([1 - f_R]d_k)^2 + (f_H h_f)^2\right\}^{\frac{1}{2}} & : \ k \neq 1 \end{cases}$$

$$\delta_k = \left\{(x_b - x_k)^2 + (h_b - h_f)^2\right\}^{\frac{1}{2}}$$

So then, using Equation(4), the extension of the string will be

$$E_k = L_f - L_u = \alpha_k + \beta_k + \gamma_k + \delta_k - L_u$$

3.6 Calculate String Tensions

Calculate the tensions in the string when fretted at fret k using (5):

$$T_k = T + EA\frac{E_k}{L_{total}}$$

3.7 Calculate Actual Frequencies

Calculate the actual frequencies produced at each fret using (1):

$$f_k = \frac{1}{2S}\left(\frac{T_k}{\rho}\right)^{\frac{1}{2}}$$

3.8 Calculate Frequency Errors

Calculate the errors between actual frequencies and required frequencies at each fret using (12):

$$\Delta F_k = 1200 \log_2\left(\frac{f_k}{F_k}\right)$$

3.9 Adjust T, h_n and h_b, and Repeat...

What we then have to do is to adjust the parameters of the model, the tension of the string, T, the nut and bridge heights, h_n and h_b, and repeat Sections 3.5 to 3.8 to minimise the frequency errors at each fret. What determines an end point to the process is really up to the player, but as a first stab we could just keep doing this until the frequency errors at all frets are reduced so that they are all below some arbitrary threshold.

3.10 Intonating a String

So how can we use this to intonate a string? Well, Figure 8 shows you how, in plain English. If you want to use the mathematics of this article, Figure 8 turns into Figure 9. f_t is the threshold for frequency errors. Get the error at all frets below f_t, and the guitar is intonated!

Figure 8: Intonation Algorithm

UNTIL you get the intonation right for the string
  UNTIL you get the open and 12^{th} frets intonated
    Adjust the tension in the string until the open string is perfect
    Adjust the bridge position of the string until the 12^{th} fret is perfect
  END UNTIL you get the open and 12^{th} frets intonated

  Adjust the nut position until the first fret is perfect
END UNTIL you get the intonation right for the string

Figure 9: Intonation Algorithm Reprise

UNTIL you get the intonation right for the string
  UNTIL $\Delta F_0 < f_t$ AND $\Delta F_{12} < f_t$
    Adjust the tension in the string until $\Delta F_0 < f_t$
    Adjust h_b until $\Delta F_{12} < f_t$
  END UNTIL $\Delta F_0 < f_t$ AND $\Delta F_{12} < f_t$

  Adjust h_n until $\Delta F_0 < f_t$
END UNTIL you get the intonation right for the string

4 Conclusion

The algorithm described in Section 3 is well suited to implentation by computer program. I have produced a spreadsheet and a Java program to implement this model.

A player can use the software to get an initial idea as to where to place the nut and the bridge for each string, given the host of constants modelling the string and the way the player frets strings.

If the string is not intonated sufficiently for the player, due to inadequacies of the model and mis-estimation of data values, the nut and bridge positions can be further adjusted to bring the string into tune. But the algorithm should take a great deal of effort out of the process by providing a good initial estimate of nut and bridge locations for each string.

5 Testing

Not done yet!

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