Steve Smith Tuition: Physics Notes

The Shell Theorem

Contents

Prerequisites

None.

Notes

None.

Document History

Date	Version	Comments
14th December 2016	1.0	Initial creation of the document.

1 Introduction

When you first learn about Newton's Law of gravity, that the size of the force F on an object of mass m is given by

$$F = \frac{GMm}{r^2}$$

you are told that the r is the distance from your object *from the center* of the massive body (of mass M and radius R). So when you do calculations, that's what you use for the distance measurement.

But the total mass M isn't all at the center of the body, is it? So why can we assume that it is?

And another intriguing question: what happens when your object (of mass m) is *inside* the massive body? There might not be much use in asking a question like this for a solid planet, but if you had gaseous object, like a star, then you could easily have that situation.

Newton considered what gravitational forces would act on a small mass m both inside and outside a large mass. To do this, he considered the large mass as being made up of spherical shells, like perfectly spherically symmetric onion rings, and thought about the gravitational effects each *shell* would have on our mass m...

2 Calculating the Gravitational Potential at a Small Mass m...

2.1 ...Outside the Shell

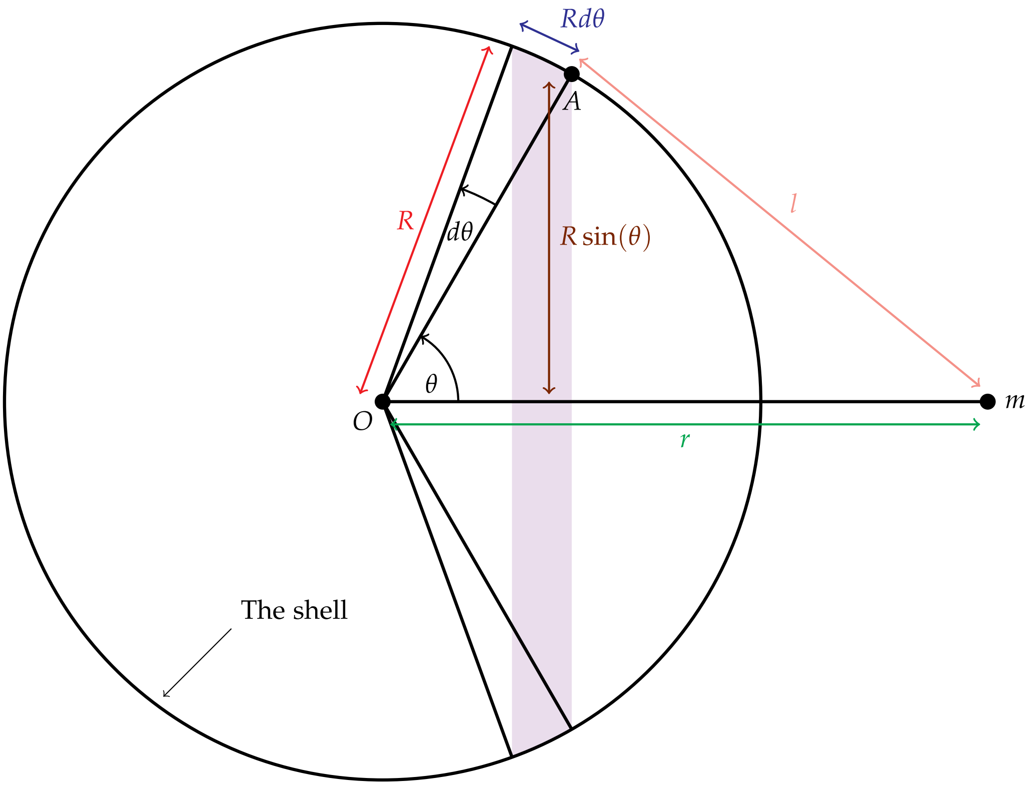

Figure 1: The Shell : 1

Figure 1 is my attempt at drawing a very thin spherical shell of matter at a distance R from the center (O) of a large spherical body of mass. So this shell is kind of like the surface of a ping-pong ball: a thin hollow shell of matter. Let the mass of the entire spherical shell be M.

The shaded zone in Figure 1 represents a very thin ring of matter in the shell, a vertical slice through the shell, located at an angle θ to the line joining O to our small mass m outside of the shell. So if you were looking at the shell from m's point of view, the shaded region would look like a *circle* of material.

I'm now going to attempt to calculate the gravitational potential at m due to this ring of material in the shell.

Now, if the thickness of this ring of material is small, then $d\theta$ is small, and the width of the ring will be approximately $R\ d\theta$.

The length of this ring of material will be $2\pi R \sin(\theta)$, as the ring is a circle of radius $R \sin(\theta)$.

That means that the surface area of this thin ring of the shell will be $2\pi R \sin(\theta) \times R\ d\theta = 2\pi R^2 \sin(\theta)\ d\theta$.

If the total mass of the shell is M, and that mass is evenly distributed over the surface area of the shell, then the mass of our thin ring will be

$$\begin{aligned}\text{mass of ring} &= \frac{M}{4\pi R^2} \times 2\pi R^2 \sin(\theta)\ d\theta \\ &= \tfrac{1}{2} M \sin(\theta)\ d\theta\end{aligned}$$

because the surface area of the whole shell is $4\pi R^2$. Now, all of the material in this ring is at a distance l from m, so the gravitational potential due to this ring of matter at m will be

$$\begin{aligned}V_{ring} &= -\frac{G \cdot \frac{1}{2} M \sin(\theta)\ d\theta}{l} \\ &= -\frac{GM \sin(\theta)\ d\theta}{2l}\end{aligned}$$

Now we can find l in terms of r and R using the cosine rule on the triangle OAm:

$$l^2 = R^2 + r^2 - 2Rr\cos(\theta)$$

If we differentiate this expression with respect to θ we will get

$$2l \frac{dl}{d\theta} = 2Rr \sin(\theta)$$

(differentiating *implicitly* on the left-hand side(!)) so that

$$\frac{dl}{Rr} = \frac{\sin(\theta)\ d\theta}{l}$$

That means the gravitational potential of the ring at m can be written

$$V_{ring} = -\frac{GM}{2Rr}\ dl$$

OK. So if we now want to find the total gravitational potential at m due to the whole shell, we integrate this expression from $l = r - R$ (the shortest distance from m to the shell) to $l = R + r$ (the distance from m to the point on the shell furthest away from m):

$$V_{shell} = -\frac{GM}{2Rr} \int_{l=r-R}^{l=R+r} dl$$

When we do this integration we get

$$\begin{aligned}V_{shell} &= -\frac{GM}{2Rr}\Big[l\Big]_{l=r-R}^{l=R+r} \\ &= -\frac{GM}{2Rr}\Big[\{R+r\} - \{r-R\}\Big] \\ &= -\frac{GM}{2Rr}\Big[R + r - r + R\Big] \\ &= -\frac{GM}{2Rr}\Big[2R\Big] \\ &= -\frac{GM}{r}\end{aligned}$$

So that

$$V_{shell} = -\frac{GM}{r} \tag{1}$$

Now Equation (1) is the same formula we would get if we assumed that the mass M of the shell *was concentrated at the center of the sphere*.

2.2 ...Inside the Shell

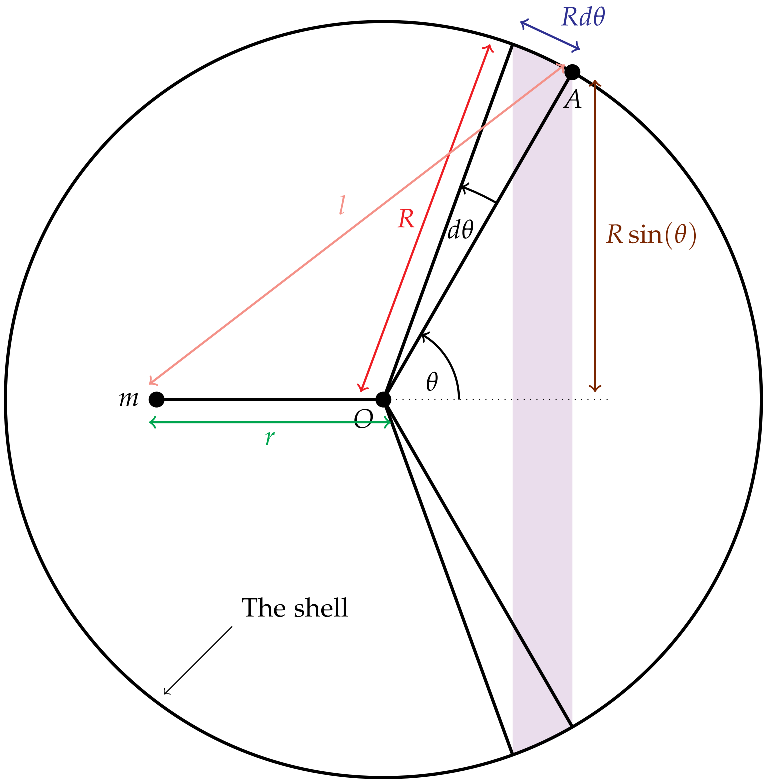

Figure 2: The Shell : 2

We can go through the same analysis as for the point mass m being outside the shell, and the only difference will turn out to be the limits on the integral. In this case the limits will be from $l = R - r$ (the shortest distance from m to the shell) to $l = R + r$ (the distance from m to the point on the shell furthest away from m):

$$V_{shell} = -\frac{GM}{2Rr} \int_{l=R-r}^{l=R+r} dl$$

When we do this integration we get

$$\begin{aligned} V_{shell} &= -\frac{GM}{2Rr} \Big[l \Big]_{l=R-r}^{l=R+r} \\ &= -\frac{GM}{2Rr} \Big[\{R+r\} - \{R-r\} \Big] \\ &= -\frac{GM}{2Rr} \Big[R+r-R+r \Big] \\ &= -\frac{GM}{2Rr} \Big[2r \Big] \\ &= -\frac{GM}{R} \end{aligned}$$

So that

$$V_{shell} = -\frac{GM}{R} \qquad (2)$$

Now Equation (2) *does not depend on* r!!! That means that the potential at m due to the shell *is the same everywhere inside the shell*, and it only depends on the parameters of the shell (its mass and its radius). This is an amazing result.

3 Calculating the Force at a Small Mass m...

So, what *force* will the mass m experience due to the mass in the shell, when it is both inside and outside of the shell?

Well, gravitational potential energy U is defined to be

$$U = mV$$

and force is defined to be

$$F = -\frac{dU}{dr}$$

3.1 ...Outside the Shell

So for a point mass m located outside the shell, at a distance r from the center of the shell, the force on m due to the shell will be

$$\begin{aligned} F &= -\frac{dU}{dr} \\ &= -m\frac{dV}{dr} \\ &= -m\frac{d}{dr}\left[-\frac{GM}{r}\right] \\ &= -m\left[\frac{GM}{r^2}\right] \\ &= -\frac{GMm}{r^2} \end{aligned}$$

Which is *exactly* the same force that m would experience if the whole of the mass M of the shell was concentrated at the center of the sphere.

3.2 ...Inside the Shell

And for a point mass m located inside the shell, at a distance r from the center of the shell, the force on m due to the shell will be

$$\begin{aligned} F &= -\frac{dU}{dr} \\ &= -m\frac{dV}{dr} \\ &= -m\frac{d}{dr}\left[-\frac{GM}{R}\right] \\ &= 0 \end{aligned}$$

Wow!!!

4 Results

Assuming a body is spherically symmetrical, so that it is made of lots of shells, in such a way that each shell has uniform mass density (but each shell can have a different mass density from all the others), then:

- for a point mass m outside of the body (outside of all the shells), then the situation is the same as if the entire mass of the body was located at the center of the sphere;
- for a point mass m inside of the body (so it is outside some of the shells, but inside others), then the gravitational force on the mass will only be due to the mass of the shells inside it. The shells outside of the mass m have no gravitational influence on m in any way.

5 Force as a Function of Distance for an Object Inside a Sphere

Let's now consider the force on an object of mass m inside a uniform (its density is the same everywhere) spherical body of mass M and radius R, as a function of distance r from the center of the body.

Firstly, since density is mass divided by volume, then the density of the body is given by

$$\rho = \frac{M}{\frac{4}{3}\pi R^3}$$

So the mass enclosed within a radius r, $M(r)$ from the center of the body will be

$$M(r) = \tfrac{4}{3}\pi\rho r^3 = \frac{M}{R^3}r^3$$

So, for our mass m at a distance r from the center of the body, the gravitational force on it will be

$$\begin{aligned} F(r) = -\frac{GM_1M_2}{R^2} &= -\frac{G \cdot \frac{M}{R^3}r^3 \cdot m}{r^2} \\ &= -\frac{GMm}{R^3}r \end{aligned}$$

which is proportional to r (see Figure 3).

$$F(r) = -\frac{GMm}{R^3}r \tag{3}$$

So, inside the body, the force of gravity will be linear, and outside the body it will drop off as an inverse square function (see Figure 3).

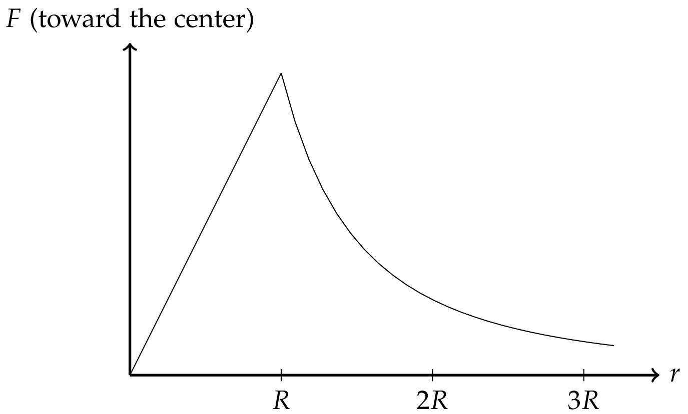

Figure 3: The Force of Gravity on an Object at a Distance r from a Large Mass of Radius R