Steve Smith Tuition: Physics Notes

Static Fields I: Gravity

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Prerequisites

(1) This document assumes that the reader can differentiate and integrate polynomials. These are topics from most AS-Level Mathematics courses. If you don't do AS-Level Mathematics, or can't remember how to differentiate and integrate polynomials, don't despair! Instead, see Ref (Smith, 2015b).

(2) You should be aware of Newton's Laws, especially $F = ma$.

(3) You should be aware of the Potential Energy equation: $E_p = mgh$[1].

(4) You should know the equation $F = \frac{mv^2}{r}$ from circular motion.

Notes

These documents about gravity and electric fields have been very difficult to write. The reason is that for A-Level students, I can't do the thing properly and give the complete vector treatment.

So I've had to lie. But only a little bit, here and there. And at least I've done it in a self-consistent manner.

So if you are a university student and you've got hold of these notes somehow, I'd be very careful about using them. Instead, I'm hoping that one day I'll write the true story. Until then, you'll have to indulge me...

Document History

Date	Version	Comments
26th December 2013	1.0	Initial creation of the document.
15th October 2014	1.1	First full version.
18th October 2014	1.2	Included a short description of the difference between uniform and radial fields.
25nd October 2014	1.3	Updating the summary to discuss the signs of the equations; updated references.
28th March 2015	1.4	Added the section showing that field lines point in the direction of decreasing potential.

[1] There are a number of different symbols used to denote potential energy. The most common (and best) is U, but most students don't encounter that symbol until after A-Level. Another common symbol used is E_p, short for $\text{Energy}_{\text{potential}}$. That's the one I'm going to use, as for A-Level students, I think it's the most intuitive.

1 Introduction

Here, we are only interested in static fields. That is, fields that don't change with time. There are three static fields that A-Level students encounter: gravitational fields, electric fields and magnetic fields. This document is only concerned with gravitational fields. For a discussion of electric fields see Ref Smith (2014b), and to find out about magnetic fields, see Ref Smith (2015a).

For a document that summarises gravitational fields, see Smith (2014a); for some example questions on gravitational fields, see Smith (2014c); for some example questions on electric fields, see Smith (2014d).

1.1 What is a Field?

In physics, a field is simply a region of space where an object of a particular kind would experience a force. What I mean by *a particular kind* is that objects experience forces because of some property that they have. For example, an object that has *mass* experiences a *gravitational* force when located in a *gravitational* field. And an object that has *charge* experiences an *electric* force when located in an *electric* field. Interestingly, gravitational fields are caused by masses, and electric fields are caused (at A-level, at least) by electric charges.

1.2 Radial and Uniform Fields

There are two kinds of gravitational or electric fields that you come across at A-Level. They are known as *radial* and $uniform$ fields.

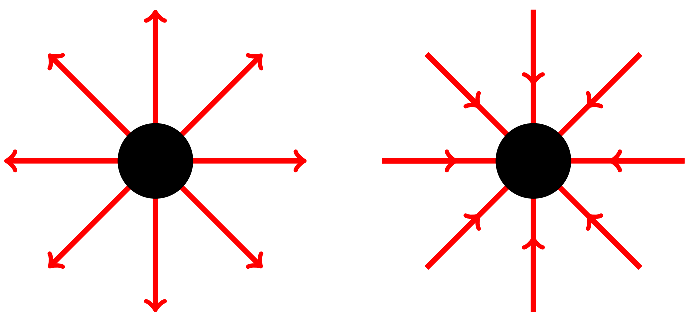
1.2.1 Radial Fields

Radial fields look like those in Figure 1. Radial fields are caused by central objects, often considered to be points. Examples of radial fields include:

- gravitational fields caused by a large central mass, such as a star or a planet;
- electric fields caused by a central charge, such as a proton or an electron.

We call these radial fields because the field lines (indicated in red in Figure 1) look like radiuses of spheres centered on the point objects.

Figure 1: Radial Fields



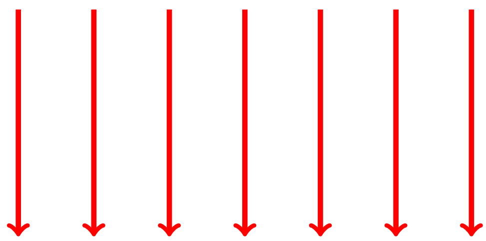
1.2.2 Uniform Fields

Uniform fields look like that in Figure 2. Uniform fields are caused by extended objects, often planes. Examples of uniform fields include:

- gravitational fields, where the scale of the situation is very small compared with the size of the object creating the field;
- electric fields caused by capacitor plates.

We call these uniform fields because the field lines (indicated in red in Figure 2) all point in the same direction, and are uniformly distributed in space (another way of saying that the field strength (see later) is the same everywhere).

Figure 2: Uniform Fields

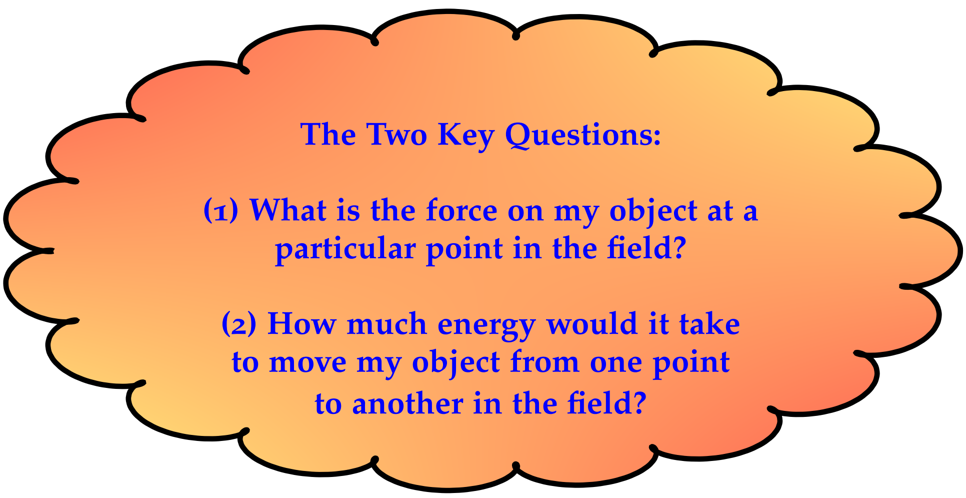


1.3 The Two Key Questions

When analysing fields, there are only two questions that we need to answer. These are:

- **What is the force on my object when it is placed at a particular point in the field?**
- **How much energy will it take to move my object from one point to another in the field?**

When you study anything to do with fields, these are the two things to have uppermost in your mind as you read. Keep asking yourself these two questions. Everything about fields boils down to these two key questions. As you read on, don't forget to keep these questions in mind. Have I emphasised that enough? You know, I'm not sure I have! What were those questions again...?



So to start answering these questions, let's begin by having a look at one of the fields we are interested in at A-Level. Let's start by having a look at radial gravitational fields...

2 The Force of Gravity

Isaac Newton (1643-1727) discovered that in the universe, every object that has mass is attracted by a force to every other object that has mass.

The formula for the *size* of the force is given by:

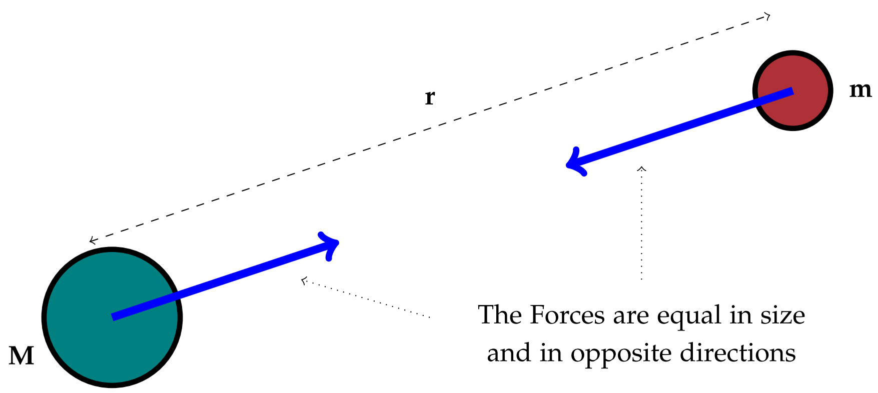
$$F = \frac{GMm}{r^2} \tag{1}$$

where M and m are the masses of two objects (units: kg) a distance r (m) apart and F is the force (in *Newtons*) between them. G is a constant to make the numbers and units right: it has a value of about 6.67×10^{-11} ($N\ m^2\ kg^{-2}$).

The way I usually think about it in problems, the upper-case M represents the mass of the object that creates the gravitational field. The lower-case m is the mass of the (small) object that experiences the force F in the field.

One interesting thing about this formula is that *it is symmetric as far as the masses are concerned*. What I mean by that is that it wouldn't matter which mass was which. You could exchange M and m, and you'd get the same formula. That means that the larger mass M would experience *exactly the same* force due the gravitational field of m as m does in the gravitational field of M. This, incidentally, is an example of Newton's Third Law of Motion.

Figure 3: Newton's Third Law of Motion



Notice in Figure 3 that r is the distance *between the centres* of the masses.

So, already, we have a way of answering one of our key questions for the case of a gravitational field. What were those questions, again?

2.1 Gravitational Force Example

Without further ado, let's have an example.

? Example 1

Let's find the force on a person of mass 50 kg due to the mass of the Earth. Let's assume for now that the person is standing on the surface of the Earth.

Example 1

Well, using the known data (see Appendix A):

- mass of Earth (M): $5.9742 \times 10^{24}\ kg$
- mass of person (m): $50\ kg$
- radius of Earth (r): $6.3744 \times 10^{6}\ m$
- gravitational constant (G): $6.67 \times 10^{-11}\ N\ m^2\ kg^{-2}$

then our equation becomes:

$$F = \frac{6.67 \times 10^{-11} \cdot 5.9742 \times 10^{24} \cdot 50}{(6.3744 \times 10^{6})^2} = 490.34\ N\ (5\ sf)$$

So the size of the force on the person is about 490 N.

2.2 Acceleration Due To Gravity

But hang on a minute, I hear you cry! I know how to calculate the force on a $50\ kg$ person due to the gravitational attraction of the Earth: it's called the person's *weight*, and it is given by Newton's Second Law formula ($F = ma$):

$$F = mg \tag{2}$$

(because g is the acceleration due to gravity) so that for a 50 kg person,

$$F = 50 \cdot 9.81 = 490.5\ N\ (4\ sf)$$

Oh wow - they're the same (well almost)!

Actually, it turns out that the formula for calculating someone's weight, (Equation 2), is just a special case of Newton's universal gravitational formula, (1).

Let's think about that for a minute. For a person of mass m on the surface of the Earth, Newton's gravitational formula (1) must agree with Newton's Second Law formula (2):

$$F = mg = \frac{GMm}{r_E^2}$$

(r_E is the radius of the Earth.)

Cancelling the m from the centre and right hand side of this equation gives:

$$g = \frac{GM}{r_E^2} \tag{3}$$

and yields a value of:

$$g = \frac{6.67 \times 10^{-11} \cdot 5.9742 \times 10^{24}}{(6.3744 \times 10^{6})^2} = 9.80679...ms^{-2}$$

which is pretty close to (in fact more accurate than!) the figure of 9.81 that we know and love.

Equation (3) also shows us that g is going to be a function of distance from the Earth's centre. It will only have the value of 9.81 when $r = r_E$ = the radius of the Earth. If you went up into the air (in a plane say) then the r that you would use in Equation (3) would be bigger than r_E, and so g would be less than it would have been on the surface. You weigh less when you fly! Neat!

3 Gravitational Field Strength

The region around the Earth where the Earth's gravity primarily acts is known as the Earth's gravitational field. And Equation (3) tells you how strong this field is at a distance r away from M. The formula $g = \frac{GM}{r^2}$ tells you the force that a 1 kg mass would experience at a distance r from the centre of the Earth:

$$F = \frac{GMm}{r^2} = \frac{GM \times 1}{r^2} = \frac{GM}{r^2} = g(r)$$

when $m = 1\ kg$. [I've also replaced g by $g(r)$ just to emphasise that g is a function of distance from the Earth's centre.]

So $g(r)$ is a measure of the force on 1 kg of mass[2] on an object a distance r from the centre of the Earth. If you knew the local value of $g(r)$ wherever an object was (i.e. its value of r), then you would know the force per unit mass on that object, and all you would have to do to find the force on the object is to multiply the mass by the local $g(r)$.

Consequently, another way to answer the first key question about the field around the earth is by having knowledge of the local value of $g(r)$. $g(r)$ is known as the field strength, or, because the field is due to a gravitational force, the gravitational field strength.

And also notice, from what we did in Section 2.2 that gravitational field strength *is the same thing as* the acceleration due to gravity!

3.1 Gravitational Field Strength Example

? Example 2

Suppose that an astronaut of mass 100 kg is in an orbit 2000 km above the Earth's surface.
(a) what is the gravitational field strength in this orbit?
(b) what is the gravitational force due to the Earth that he experiences?

Example 2

(a) In order to calculate the local gravitational field strength, we use the formula:

$$g(r) = \frac{GM}{r^2}$$

Now for an orbit at a height of 2000 km (= $2 \times 10^6\ m$), the r we need will be $r = r_E + 2 \times 10^6 = 6.3744 \times 10^6 + 2 \times 10^6 = 8.3744 \times 10^6\ m$.
So the local gravitational field strength at the orbit distance will be:

$$\begin{aligned} g(r) &= \frac{6.67 \times 10^{-11} \cdot 5.9742 \times 10^{24}}{(8.3744 \times 10^6)^2} \\ &= 5.68\ ms^{-2}\ (3\ sf) \end{aligned}$$

(b) Now we know the local gravitational field strength at the distance of the orbit, it's easy to work out the force on the astronaut. It will be:

$$F = mg = 100 \cdot 5.68 = 568\ N\ (3\ sf)$$

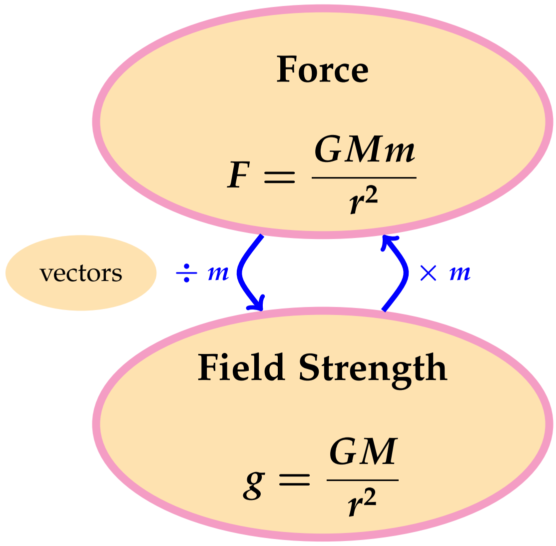
From this I hope that you can see that the field strength is a property of position (only!) around a large mass M, such as the Earth. The nearer you are to the mass, the stronger the field and the greater the force on you; the further away you are, the weaker the field, and the weaker the force.

[2] Physicists call that the *force per unit mass*.

4 The Relationship Between Force and Field Strength: the Story So Far...

Here's a picture of what we've learned so far about the relationship between force and field strength for a gravitational field.

Figure 4: Relationship Between Force and Field Strength



Hopefully, then, you can now see that there are in fact *two* ways of answering our first key question: "What is the force on my object". These are:

- **Find the force directly using $F = \frac{GMm}{r^2}$, or...**
- **Find the field strength using $g = \frac{GM}{r^2}$, and then use $F = mg$ to give you the force on your object.**

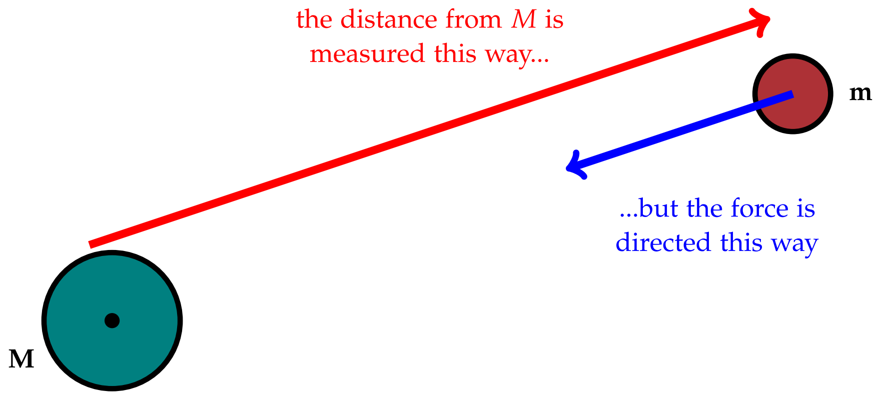
And of course, from Figure 4, if you knew the force on an object of known mass, you can get the field strength of the field where the object is by dividing the force by the mass.

5 ...But Force is a Vector

Now we haven't got very far into our story, but already we have an issue.

Force is a *vector* quantity, which means that it has a *direction* associated with it, as well as a size. So how can we take that into account? Have another look at our two masses. Have a look at the force on the smaller mass m due to the larger mass M. See Figure 5.

Figure 5: The Direction of the Force



The force on m is in the *opposite* direction to the way that the distance from the center of M is measured. That's awkward. Now for A-Level, where we don't want the full vector treatment, the way we tackle this problem is to introduce a negative sign into the force formula:

$$F = -\frac{GMm}{r^2} \tag{4}$$

to indicate that the direction of the force is in the opposite direction to the way distance is measured.

Now you may think that the negative sign is kind of irrelevant. We all know that gravitational forces are attractive, so we kind of know which way the force will act. Well, that's true. But when we come to look at electric fields, fields that act on charges, we find that forces can go either way, depending on the sign of the charges involved. Forces can be repulsive as well as attractive. So we definitely need to know directions then. And the negative sign is needed here to make the gravitational field equations consistent with the electric field equations. More about this later.

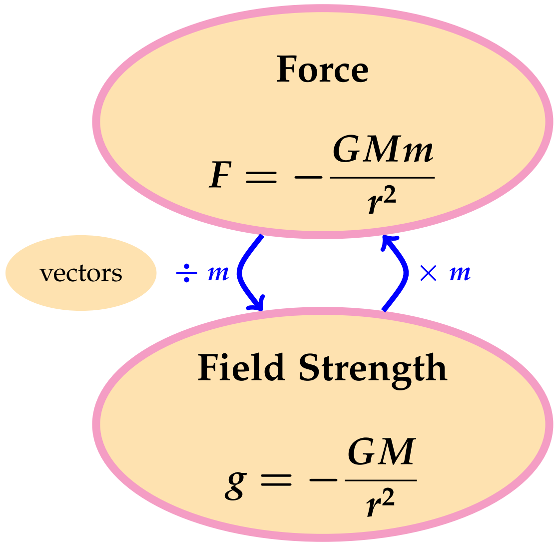
If you like, you can think of the signs of the forces as being a kind of convention. And the convention turns out to be:

OK, so how does that change things? Well, not too much, actually. The only thing that we have to do is to insert a negative sign into the force formula.

Oh no - that's not quite the only thing. Remember from Figure 4 that we can get the field strength by dividing force by mass? Well, since the force formula now has a negative sign in it, then so will the field strength formula. And that's because *field strength is also a vector*.

So, the real relationship between force and field strength for a gravitational field is the one that's shown in Figure 6.

Figure 6: The Real Relationship Between Force and Field Strength



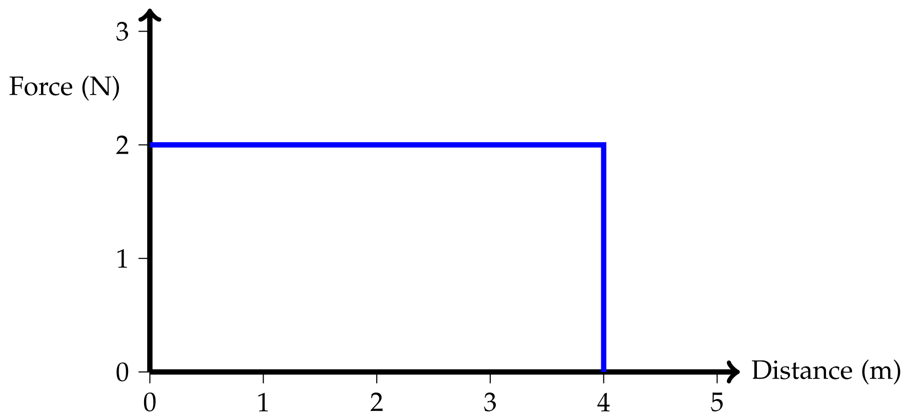
6 Gravitational Potential Energy

Now it's at this point where we start to answer the second key question: "How much energy will it take to move an object from one point in a field to another?"

6.1 Area Under the Force-Distance Graph

Let's begin by looking at a graph of Force v Distance, such as the one shown in Figure 7. Then the area under the graph would represent Force × Distance. For this particular Force v Distance graph, the area would be $2\ N \times 4\ m = 8\ Nm$.

Figure 7: A Force v Distance Graph



But we know from mechanics that

$$\textit{Force} \times \textit{Distance} \equiv \textit{Energy}$$

which means that the area under a Force v Distance graph must represent some kind of energy. What kind of energy?

Let's look at the Force v Distance graph for the gravitational force, and see if we can make some kind of sense of this (see Figure 8).

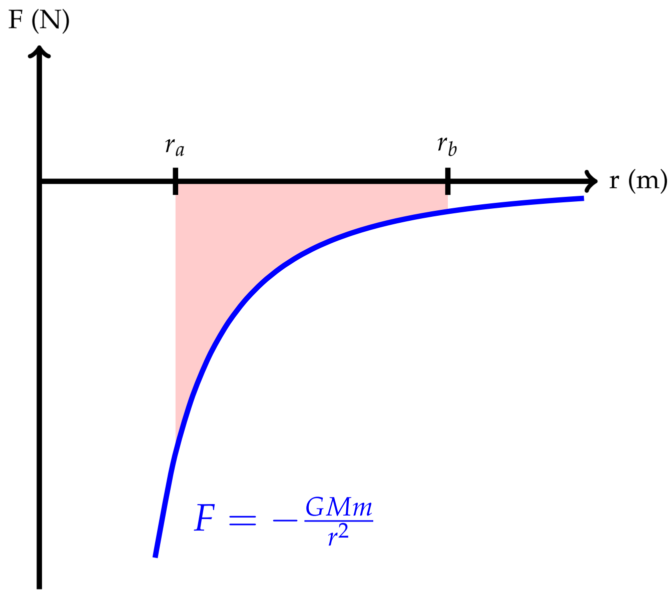
In Figure 8 the shaded region represents an area between two values of r: from $r = r_a$ to $r = r_b$. So is this area the energy needed to move an object from a distance r_a from the origin of the force to a distance r_b away from it? YES! That makes sense: because the mass M causing the field pulls on our little mass m, then it will take effort to move m away from M. It will take *energy* to do this.

That means that this energy must be associated only with the *position* of an object. Haven't we come across that idea somewhere before? Isn't that what they call *potential* energy? YES, YES, YES!!

And in this case, as the force is gravitational, this is gravitational potential energy.

So, how do we find the area under this curve? We need calculus to help us here because the force function is a curve (rather than a straight line, as in Figure 7). And if you are a budding A-Level mathematician, you might have spotted what we need to do here: we will need to *integrate* the force function with respect to r. And there's an additional complication here: because our force function is negative, the area obtained by integrating will be negative. But energies can't be negative, so we need to take the negative of this integral to find the real energy! Phew!!

Figure 8: Force v Distance For Gravity



The area under this curve will be equal to:

$$Area = -\int_{r=r_a}^{r=r_b} -\frac{GMm}{r^2}\, dr$$

or, simplifying the signs,

$$Area = \int_{r=r_a}^{r=r_b} \frac{GMm}{r^2}\, dr$$

Now because G, M and m are all constants (and not functions of r), then this integral becomes:

$$Area = GMm\int_{r=r_a}^{r=r_b} \frac{1}{r^2}\, dr$$

which, by the normal rules of polynomial integration, is

$$Area = E_p = -\frac{GMm}{r} + C \qquad (5)$$

where I've written the area as E_p because it represents a potential energy, and C is the integration constant (don't forget the constant when you integrate!).

Now we need to talk about this constant. Remember that from integration theory, this constant can have any value, *positive or negative*. So, can we just choose what value we want for this constant? It turns out that the answer to this question is essentially *yes*, although physicists have chosen a value, by convention, for it.

Quite a lot of the time, as we will see, we are only interested in the $difference$ between one potential energy and another. And if we take one potential energy away from another, then the constant C is irrelevant, as it will canel out.

But when we are interested in the value of a single potential energy, we will need to know the value of C. Believe it or not, to simplify matters as much as possible, it is conventional to take the potential energy of a system to be *zero* when the component parts of the system are at infinite separation from one another (so the force between them is zero). This convention is quite arbitrary, but has the convenient consequence that when the distance between the masses is infinity, then $-\frac{GMm}{r}$ will be zero, and so the constant of integration C will be zero, too.

But this has the unfortunate consequence of making potential energy values in fields negative. This can be confusing!

So: all gravitational potential energies are negative(!), and they get smaller (i.e. more negative) the closer you are to the centre of the force. The further out from the centre of the force, the bigger (i.e. less negative) they get, until the potential energy will be zero at infinity (i.e. when completely removed from the influence of the central force).

6.2 Change in Gravitational Potential Energy

Going back to Figure 8 for a moment, we can now find the energy needed to move the mass from $r = r_a$ to $r = r_b$ in the gravitational field. It will be equal to

$$\begin{aligned} Area \;=\; E_p \;&=\; GMm \int_{r=r_a}^{r=r_b} \frac{1}{r^2}\, dr \\ &=\; GMm \left[-\frac{1}{r} \right]_{r=r_a}^{r=r_b} \\ &=\; GMm \left[\frac{1}{r_a} - \frac{1}{r_b} \right] \end{aligned}$$

This actually ties up with stuff we've done before, because when we did potential energy in mechanics, we used the equation

$$E_p \;=\; mgh \tag{6}$$

where h is the height above some surface. The higher you are (i.e. further from the centre of the earth), the greater the potential energy.

So how do we tie up equations (5) and (6)? Well it turns out that (6) is just a special case of (5)!

To see this, let's say that we wanted to calculate the potential energy increase when we move from a distance r_E to a distance $r_E + h$. (Remember that r_E is the radius of the earth.) This should turn out to be mgh, shouldn't it?

Well, doing the integration, we get:

$$\begin{aligned} E_p \;&=\; GMm \int_{r=r_E}^{r=r_E+h} \frac{1}{r^2}\, dr \\ &=\; GMm \left[-\frac{1}{r} \right]_{r=r_E}^{r=r_E+h} \\ &=\; GMm \left[\frac{1}{r_E} - \frac{1}{r_E + h} \right] \\ &=\; GMm \left[\frac{h}{r_E(r_E + h)} \right] \end{aligned}$$

And because $r_E + h \approx r_E$ if h is very small compared to r_E, then this will be almost exactly

$$E_p \;=\; \frac{GMmh}{r_E^2}$$

But using (3) we get

$$E_p \;=\; \frac{GM}{r_E^2} mh \;=\; gmh$$

And so the two forms for the potential energy calculation do agree! But only when h is very small compared to r_E, and also when the location of the experiment is close to the surface of the earth, so that the specific value of g at the surface of the Earth applies! [Note that both of these conditions applied when we did our study of potential energy in mechanics!]

6.3 Gravitational Potential Energy Examples

6.3.1 A Straightforward Potential Energy Calculation

As a first example of the use of potential energy, consider the following problem. This problem highlights the most common way that gravitational potential energy is calculated and used.

? Example 3

How much energy will it take to move an astronaut of mass 100 kg from an orbit 2000 km above the Earth's surface to an orbit 4000 km above the Earth's surface?

Example 3

Well, to calculate the energy it takes to move the astronaut from the 2000 km orbit to the 4000 km orbit, we find the potential energies at the two orbits, and take the difference between them:

$$\begin{aligned}\text{(at 2000 km) } E_p &= -\frac{GMm}{r} \\ &= -\frac{6.67 \times 10^{-11} \cdot 5.9742 \times 10^{24} \cdot 100}{6.3744 \times 10^{6} + 2000 \times 10^{3}} \\ &= -4.758 \times 10^{9}\ J\end{aligned}$$

$$\begin{aligned}\text{(and, at 4000 km) } E_p &= -\frac{GMm}{r} \\ &= -\frac{6.67 \times 10^{-11} \cdot 5.9742 \times 10^{24} \cdot 100}{6.3744 \times 10^{6} + 4000 \times 10^{3}} \\ &= -3.841 \times 10^{9}\ J\end{aligned}$$

And so all we need to do to find the energy it takes is to subtract one from the other:

$$\begin{aligned}\Delta E_p &= E_p \text{ at 4000 km} - E_p \text{ at 2000 km} \\ &= -3.841 \times 10^{9} - (-4.758 \times 10^{9}) \\ &= 9.17 \times 10^{8}\ J\ (3\ sf)\end{aligned}$$

6.3.2 A Much More Interesting Example!

And as another example of the use of potential energy, consider the following situation. This highlights one of the significant uses of negative potential energies!

? Example 4

Discuss the potential and kinetic energies of a satellite in a circular orbit around a planet.

Example 4

A satellite in orbit around a planet will have both potential energy (due to its position in the field), and kinetic energy (due to its orbital speed). Following our convention, the potential energy will be negative and the kinetic energy will be positive. This will mean that if the total energy (potential + kinetic) is less than zero, the satellite will be constrained by the gravitational force of the planet, because potential energy predominates; if the total energy is positive, the satellite will escape the planet's gravity, as kinetic energy predominates.

If the satellite was in a circular orbit, then the force needed to keep it moving in a circle is $F = \frac{mv^2}{r}$, and this force will be provided by the gravitational attraction to the planet. So:

$$\frac{mv^2}{r} = \frac{GMm}{r^2}$$

and so the kinetic energy of the satellite will be given by:

$$\frac{mv^2}{2} = \frac{GMm}{2r}$$

This means that the satellite will have a total energy given by:

$$\begin{aligned} \textit{Total energy} &= \textit{kinetic energy} + \textit{potential energy} \\ &= \frac{GMm}{2r} - \frac{GMm}{r} \\ &= -\frac{GMm}{2r} \end{aligned}$$

which is negative, showing that the satellite will be bound to the planet by gravity (as you would expect if it was in a circular orbit!).

6.4 Differentiating the Gravitational Potential Energy Function

If you were completely unconvinced by my argument as to why we need to take the negative integral of force to get potential energy in Section 6.1, maybe this section might help. Here, we look at the thing from the other way round: how to get from the potential energy to the force.

If we have to $-$*integrate* force to get potential energy, then we will have to $-$*differentiate* potential energy to get force. Since

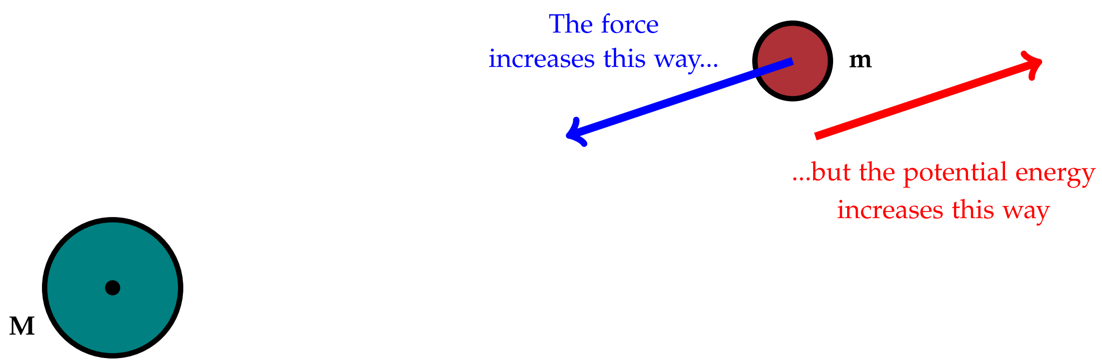
$$E_p = -\int F\,dr$$

then

$$F = -\frac{dE_p}{dr}$$

So another way to see why there should be a negative sign on these equations is to have a look at the directions of things again. Remember that force is a vector? Well, check out Figure 9.

Figure 9: The Relationship Between Force and Potential Energy



Because force increases in the *opposite* direction to the way that potential energy increases, we need the negative signs to represent this.

6.5 Gravitational Potential Energy: Key Points

Gravitational potential energy is the energy of a mass m purely due to its position in a gravitational field;

For a mass m, at a distance r from M its value is:

$$E_p = -\frac{GMm}{r}$$

Gravitational potential energies are negative! But...

...the energy required to move a mass to infinity will be positive

Potential energy is just the area under a force $\times$ distance graph, so...

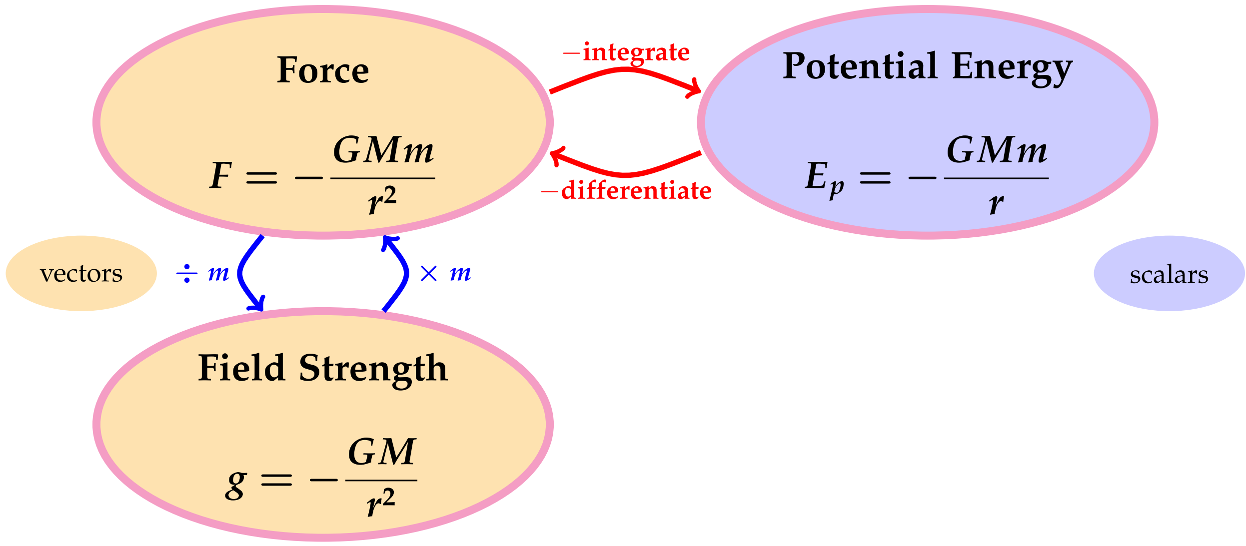
(a) $-$integrate force (with respect to distance) to get potential energy;

(b) $-$differentiate potential energy (with respect to distance) to get force.

7 Introducing Gravitational Potential Energy into the Grand Scheme

Now we can introduce the potential energy into our picture of the relationships between the important quantities involved with fields.

Figure 10: The Relationship Between Force, Field Strength and Potential Energy



8 Gravitational Potential

Potential is defined so that potential energy and potential have the same relationship as do force and field strength. Just as field strength is the force per unit mass on an object, well, potential is just the potential energy per unit mass of an object. And physicists have given potential the symbol V. Because it is potential energy per unit mass, its units will be $J\ kg^{-1}$.

Remember that to get the force on an object of mass m if you know the local field strength:

$$F = gm$$

So to get the potential energy of a mass m if we know the potential at a point, then we just use:

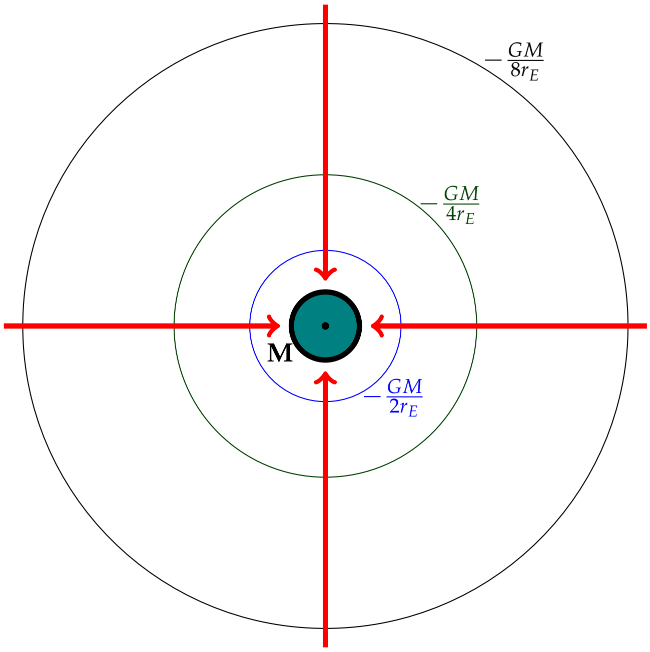
$$E_p = Vm$$

So the formula for gravitational potential will be:

$$V = \frac{E_p}{m} = -\frac{GMm}{rm} = -\frac{GM}{r} \tag{7}$$

8.1 Field Lines Point in the Direction of *Decreasing* Potential

Figure 11: Field Lines and Potential



In Figure 11 is a planet, M, of radius r_E. On the figure are shown some field lines, in red, and some equipotential lines (an equipotential line is one where the potential is the same all along the line).

Using the potential formula (7), I have marked the potentials on the equipotential lines. Hopefully you can see that because of the $-$ sign, the potentials are *increasing* as you go away from the planet.

This bring up another interesting point about this field: *the field lines of the field around M point in the direction of decreasing potential*. We will discover that this occurs with *all* fields.

And in the case of a gravitational field, this is equivalent to saying "gravity always pulls things *down*hill".

8.2 Gravitational Potential Examples

8.2.1 A Straightforward Calculation

To show how gravitational potential can be calculated and used, consider the following situation.

? Example 5

Suppose that an astronaut of mass 100 kg is in an orbit 2000 km above the Earth's surface.
(a) what is the gravitational potential in this orbit?

It is required that that astronaut be moved to an orbit 4000 km above the Earth's surface.
(b) what is the gravitational potential in this new orbit?

Example 5

(a) In order to calculate the local gravitational potental, we use the formula:

$$V = -\frac{GM}{r}$$

Now for an orbit at a height of 2000 km (= 2×10^6 m), the r we need will be $r = r_E + 2 \times 10^6 = 6.3744 \times 10^6 + 2 \times 10^6 = 8.3744 \times 10^6$ m, so the local gravitational potential at the orbit distance is:

$$\begin{aligned} V &= -\frac{6.67 \times 10^{-11} \cdot 5.9742 \times 10^{24}}{8.3744 \times 10^6} \\ &= -4.758 \times 10^7 \ J\, kg^{-1} \ (4\, sf) \end{aligned}$$

(b) Now for an orbit at a height of 4000 km (= 4×10^6 m), the r will be $r = r_E + 4 \times 10^6 = 6.3744 \times 10^6 + 4 \times 10^6 = 10.3744 \times 10^6$ m, so the local gravitational potential at the orbit distance is:

$$\begin{aligned} V &= -\frac{6.67 \times 10^{-11} \cdot 5.9742 \times 10^{24}}{10.3744 \times 10^6} \\ &= -3.841 \times 10^7 \ J\, kg^{-1} \ (4\, sf) \end{aligned}$$

8.2.2 Potential Energy from Potential

But now, consider this:

? Example 6

How much energy will it take to move the astronaut from an orbit 2000 km above the Earth's surface to an orbit 4000 km above the Earth's surface?

Example 6

Well, to calculate the energy it takes to move the astronaut from the 2000 km orbit to the 4000 km orbit, we find the *potential difference* between the two orbits. We can do this by using the results from the previous example:

$$\begin{aligned} \Delta V &= (-3.841 \times 10^7) - (-4.758 \times 10^7) \\ &= 9.17 \times 10^6 \ J\, kg^{-1} \ (3\, sf) \end{aligned}$$

And so all we need to do to find the energy it takes is to multiply this potential difference by the mass of the astronaut:

$$\begin{aligned} \Delta E &= 9.17 \times 10^6 \times 100 \\ &= 9.17 \times 10^8 \ J \ (3\, sf) \end{aligned}$$

8.3 Two Ways of Finding the Energy

Hopefully, then, you can now see that there are in fact *two* ways of answering our second key question: "What is the energy it takes to move my object from one place in the field to another". These are:

- **Find the difference in potential energies between the two points directly using the formula $E_p = -\frac{GMm}{r}$ at each point, or...**
- **Find the potential difference between the two points using $V = -\frac{GM}{r}$ at each point, and then use $E_p = Vm$ to get the energy.**

8.4 Differentiating the Potential Function

Now, remember that we had to −integrate force to get potential energy? And remember that potential energy and potential have the same relationship as do force and field strength? Well that means that we have to −integrate field strength to get potential!

$$V = -\int g \, dr$$

And so we have to −differentiate potential to get field strength:

$$g = -\frac{dV}{dr}$$

8.5 Gravitational Potential: Key Points

Gravitational potential is the potential energy per unit mass of an object due only to its position in a gravitational field;

For a point at a distance r from M its value is:

$$V = -\frac{GM}{r}$$

Gravitational potentials are negative!

Potential is just the area under a field strength $\times$ distance graph, so...

(a) $-$integrate field strength (with respect to distance) to get potential;

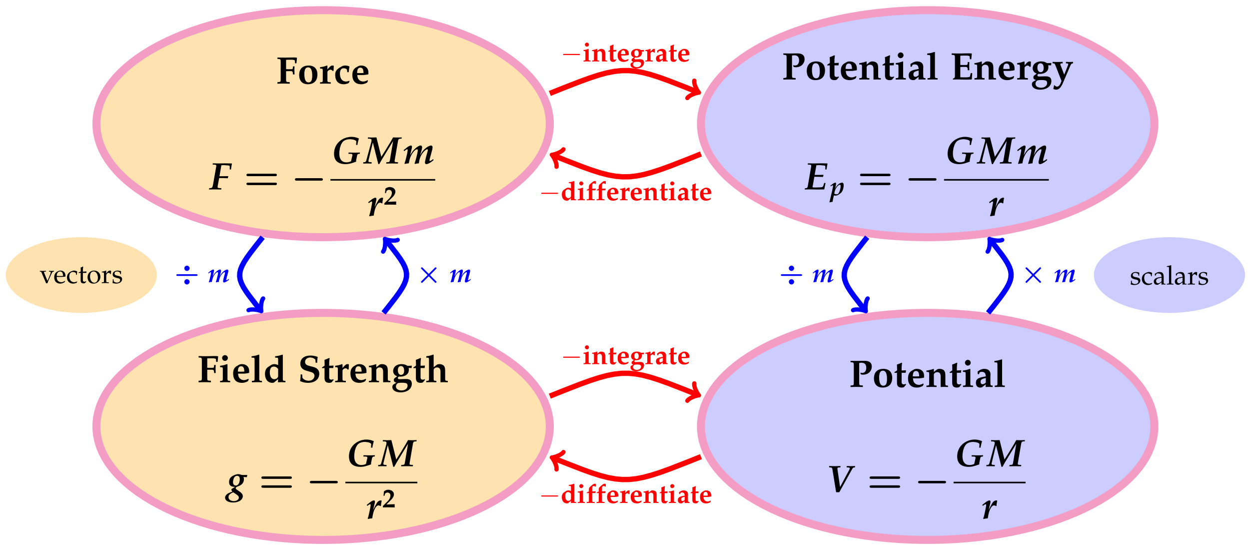
(b) $-$differentiate potential (with respect to distance) to get field strength.

Field lines point in the direction of decreasing potential

9 Gravity: Summary

We can now include the final piece of the jigsaw into our grand scheme. Have a look at Figure 12. **You can sum *everything* up by this picture.**

Figure 12: Summary of Gravitational Fields



In order to answer our two key questions, you use

- The left-hand pair of quantities (force or field strength) to determine the force on an object with mass m at a distance r from a large mass M;
- The right-hand pair of quantities (potential energy or potential) to determine the energy required to move an object with mass m at a distance r_a from a large mass M to a distance r_b from M, either by finding the difference in potential energies at the two points directly, or by finding the difference in potentials at the two points, and multiplying by the mass of the object.

One very important thing to add: if you take out the equations from the bubbles in Figure 12, and just look at the relationship between the quantities, **that relationship is the same for all fields**. That's due to the definitions of the quantities.

So if you want to understand fields, then just remember this picture. The neat thing is that because this relationship of the four key quantities is the same for all fields, all we need is the equation of one of the quantities. Then we can work out all the rest!!

One final point.

Since force and field strength are *vectors*, the *sign* in the equation indicates *direction*. Since potential energy and potential are *scalars*, the *sign* in the equation indicates *size*. So, quite often we can ignore the sign in force and field strength calculations, but we must *never* ignore the signs in potential energy and potential calculations!

10 Appendices

A Useful Data

Quantity	Value
Earth - Moon Distance	3.844×10^{8} m
Gravitational Constant	6.673×10^{-11} m^3 kg^{-1} s^{-2}
Mass of the Earth	5.972×10^{24} kg
Radius of the Earth	6.371×10^{6} m
Mass of the Moon	7.347×10^{22} kg
Gravitational Field Strength at the surface of the Earth	9.81 ms^{-2}
k	8.9876×10^{9} m^3 kg s^{-4} A^{-2}
ϵ	8.8542×10^{-12} m^{-3} kg^{-1} s^4 A^2

References

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