Steve Smith Tuition: Physics Notes

Springs, Hooke's Law and Young's Modulus

Contents

Prerequisites

Before reading this docment, you should be familiar with Hooke's Law and Young's Modulus.

Notes

None.

Document History

Date	Version	Comments
30th November 2014	1.0	Initial creation of the document.

1 Quick Recap of Hooke's Law and Young's Modulus

Hookes's Law and Young's Modulus are ideas that came out of the study of how materials stretch.

1.1 Hooke's Law

Robert Hooke (1635 - 1703) discovered that if you pull on something, it stretches. And if you don't pull on it too much, the amount it stretches is proportional to the amount you pull on it. Mathematically, if F is the force you pull on the thing, and e is the amount the thing stretches (the *extension*), then

$$F = ke \tag{1}$$

where k is known these days as the *spring constant*.

And that's Hooke's Law.

The spring constant k is also known as the *stiffness* constant because it gives you an idea of how stiff something is. In other words, how much force you would have to pull on the thing to stretch it a certain amount.

1.2 Young's Modulus

Thomas Young (1773 - 1829) took this one step further by trying to attach a number (the *Young's Modulus*) to each *material*. He realised that if you took two different samples of the same material, for example a pencil (with no lead!) and a tree trunk, then the amount of force that you would need to stretch each thing will depend on the dimensions of the objects. Even if they were made of the same material.

So he tried to devise a test to see how stiff a material (for example, wood) is. That test would have to take the dimensions of the object tested into account, so that differently sized objects of the same material would have the same stiffness.

And here's Young's fair test of stiffness:

$$E = \frac{Fl}{eA} \tag{2}$$

the Young's Modulus E being given by the amount of force F applied to the object multiplied by the length of the unstretched object l divided by the product of the extension e and the cross-sectional area of the object A.

Using this formula, it is possible to find a number for the stiffness of any material, using any specimen of that material. And so now we can compare the stiffnesses of different materials by comparing their Young's Moduli. And that's useful when you want to build things like bridges and skyscrapers.

1.3 The Connection Between Hooke's Law and Young's Modulus

This connection is a very simple one. If we put Equation (2) in the form of Equation (1), then we can find a connection between Hooke's spring constant and Young's Modulus. I'll show you what I mean. From

$$E = \frac{Fl}{eA}$$

then if we multiply both sides by eA we get

$$Fl = EeA$$

then if we divide both sides by l we get

$$F = \frac{EeA}{l}$$

Now, we can write the above equation like this:

$$F = \left(\frac{EA}{l}\right) e$$

So, comparing that to Hooke's Law, we obtain the relationship between Hooke's spring constant k and Young's Modulus E:

$$\begin{aligned} F &= \frac{EA}{l}\, e \\ F &= k\, e \end{aligned}$$

so

$$k = \frac{EA}{l} \tag{3}$$

2 Springs

Now a spring is just a piece of material that stretches (or compresses) by a significant amount when you apply a force to it. And when you take the force off, the spring returns to its natural length. That is, if the force you applied wasn't so big that it damages the spring.

So, when you apply a force to a spring, it stretches according to Hooke's Law (see Figure 1). When there

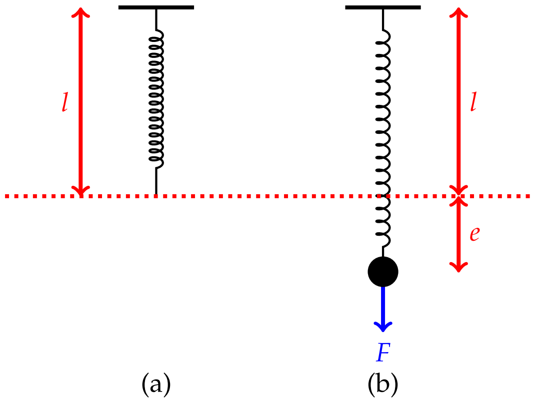

Figure 1: A Single Spring

is no force applied to the spring, it remains at its natural length, l (see Figure 1, (a)). When you apply a force F to the spring, by hanging a weight on the end of it, for example, it stretches by an amount e (see Figure 1, (b)).

And of course, the force F is related to the spring's extension e by

$$F = ke$$

where k is the spring constant.

2.1 Springs in Series and Parallel

OK so far, I hope. But, what if you were to join springs together in various combinations? What happens then?

2.1.1 Springs in Series

Springs are said to be *in series* when they are arranged as in Figure 2. In Figure 2 we have two springs in

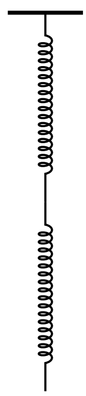

Figure 2: Two Springs In Series : 1

series. Each spring will have it's own natural length, and it's own spring constant. The question now is,

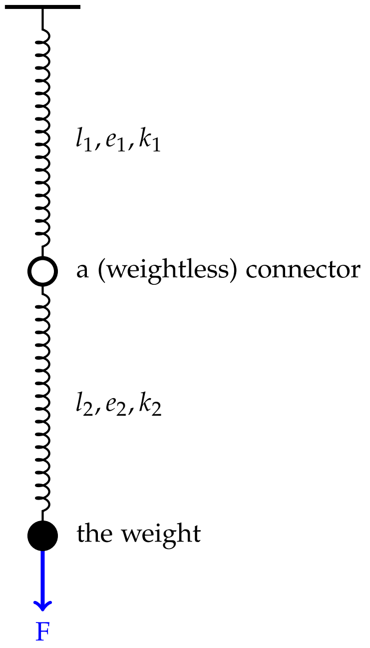

Figure 3: Two Springs In Series : 2

what would happen when you hang a weight on the end of the spring combination (See Figure 3)? Now in this case, it is reasonable to assume that both springs will stretch, but by how much?

Let's assume that the top spring has a natural length l_1, a spring constant k_1, and when we apply the force F it stretches by an amount e_1. And let's assume that the bottom spring has a natural length l_2, a spring constant k_2, and when we apply the force F it stretches by an amount e_2.

Now we analyse what's happening to each spring by drawing what are called *free body diagrams* for different objects in our little system. Free body diagrams are diagrams showing all the external forces that act on an object.

So, let's start with the weight. When the force F is applied to the spring combination, the bottom spring extends by an amount e_2. That means then that this spring will be pulling back on the weight with a *restoring* force of size

$$F_2 = k_2 e_2$$

and this force will be directed *upward* because it is resisting the stretching of the spring by the weight. See Figure 4. Now because the bottom spring is in *equilibrium* (which is a fancy way of saying that the forces

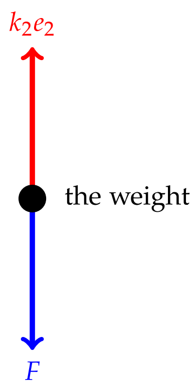

Figure 4: Forces on the Weight

on it must balance), then

$$F = k_2 e_2$$

OK, now let's have a look at the forces on the connector - the object I've placed in between the two springs. When the force F is applied to the spring combination, the top spring extends by an amount e_1. That means then that this spring will be pulling back on the connector with a *restoring* force of size

$$F_1 = k_1 e_1$$

and this force will be directed *upward* because it is resisting the stretching of the spring by the weight. But the force pulling the connector *down* will be the force exerted by the bottom spring. In other words, $k_2 e_2$ (see Figure 5). Now because the top spring is in *equilibrium* (which is a fancy way of saying that the forces

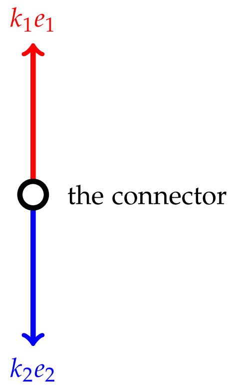

Figure 5: Forces on the Connector

on it must balance), then

$$k_1 e_1 = k_2 e_2$$

and so

$$k_1 e_1 = k_2 e_2 = F$$

So when springs are in series, the force pulling them is effectively acting *on every spring* in the series. Each individual spring will extend as if the force was acting on it alone.

Now you could think of the series of springs as one large spring. If we do that with our example, we will see that the total extension e of the entire spring will be

$$e = e_1 + e_2 = \frac{F}{k_1} + \frac{F}{k_2}$$

or, in other words

$$e = \left(\frac{1}{k_1} + \frac{1}{k_2} \right) F$$

But from Hooke's Law, if k is the spring constant of the whole spring, then

$$e = \frac{1}{k} F$$

so we can see that the spring constant of the whole spring, k will be related to the spring constants of the individual springs, k_1 and k_2 by

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$$

which reminds me of the formula for calculating the total resistance of resistors in *parallel*. Or indeed, the total capacitance of capacitors in *series*!

2.1.2 Springs in Parallel

Springs in parallel are shown in Figure 6. Each of these springs has the same natural length l, but let's

Figure 6: Springs in Parallel : 1

say that the spring on the left has spring constant k_1 and the spring on the right has spring constant k_2. If we put a force F on the combination of springs (Figure 7), again by simply hanging a weight on the combination, *and we ensure that the bar through the weight remains horizontal* so that the arrangement doesn't buckle, then both springs will extend by the same amount, e say. Since both springs have extended, they

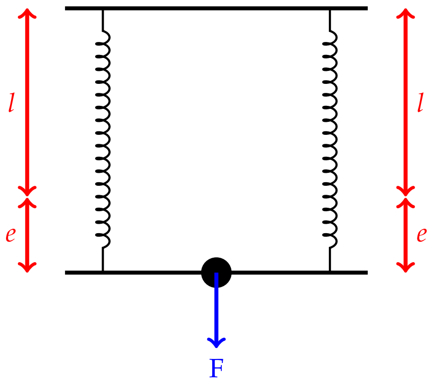

Figure 7: Springs in Parallel : 2

will exert a *restoring* force *up* on the weight. So if we do that free body force diagram thing-a-ma-jiggy on the weight we will get Figure 8. Now since the weight that we hang on this spring combination must be

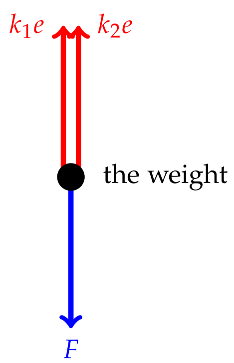

Figure 8: Free Body Force Diagram for Springs in Parallel

in equilibrium (what does that mean, again?), then

$$F = k_1 e + k_2 e$$

Or,

$$F = (k_1 + k_2)e$$

Which means that this time, the combination of springs behaves like a single spring of spring constant $k = k_1 + k_2$. Which reminds me of...

3 Applying Hooke's Law and Young's Modulus to a String of Atoms

Imagine a vertical column of spherical atoms in a material, each having a radius r, and each atom just touches its neighbours so that the distance between each pair of neighbouring atoms is d $(= 2r)$. See Figure 9. There are bonds between atoms, holding the atoms together, but not too close. These bonds can

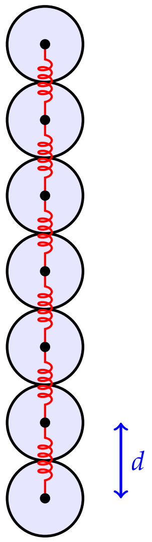

Figure 9: Vertical String of Atoms

be modelled by springs.

So let's say that we held one end of this string of atoms fixed (by attaching it to the ceiling), and applied a force F to the other end. See Figure 10. Now if we had n atoms in this chain, then the total length of the

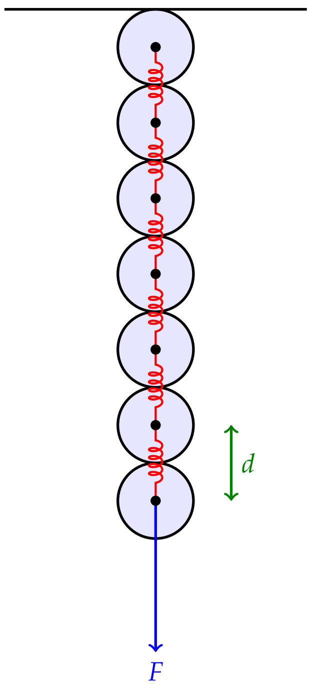

Figure 10: Vertical String of Atoms With an Applied Force

chain would be nd. And if you looked vertically down on the top of the chain of atoms (so they looked like circles) the cross-sectional area of the chain of atoms would be

$$A = \pi \left(\frac{d}{2}\right)^2 = \frac{\pi}{4}d^2$$

Instead of using this value of $\frac{\pi}{4}d^2$ as the cross-sectional area of the chain, I'm just going to use d^2. That's because this chain of atoms will not be on its own in the material; there will be many other vertical chains of atoms like this, and if all the atoms in the material were in a body-centered cublic structure, so that each atom is at the corner of a cube, then the actual effective cross-sectional area of a single vertical chain of atoms will be d^2 as d is the length of each edge of the cube. Anyway you think of this, let's say that

$$A \approx d^2$$

which means that A is approximately equal to d^2.

Now this chain of atoms is the same kind of thing as we had in Section 2.1.1. The bonds between the atoms are in series, so the springs in series formula must apply. Let the spring constant of each bond be k_b, and the spring constant of the whole chain be k. Applying the spring in series formula, then

$$\begin{aligned} \frac{1}{k} &= \frac{1}{k_b} + \frac{1}{k_b} + \frac{1}{k_b} + \cdots + \frac{1}{k_b} \\ &= \frac{n}{k_b} \end{aligned} \tag{4}$$

since there are n springs (bonds)[1].

Now in Section 1.3 we discovered that there is a connection between Young's Modulus E and the spring constant of a material, k:

$$k = \frac{EA}{l}$$

Applying this formula, we get

$$\frac{1}{k} = \frac{l}{EA} \approx \frac{nd}{Ed^2} = \frac{n}{Ed}$$

since the length of the chain is nd and the cross-sectional area is about d^2. But, using Equation (4), then

$$\frac{1}{k} = \frac{n}{k_b}$$

so

$$\frac{n}{Ed} \approx \frac{n}{k_b}$$

so that

$$E \approx \frac{k_b}{d}$$

Phew!

[1] Actually, there are only $n-1$ springs (bonds) between n atoms, but because we are talking atoms here, and the value of n will be really large, there is effectively no difference between n and $n-1$.