How high does an aircraft have to be to be visible to a radar?

Steve Smith

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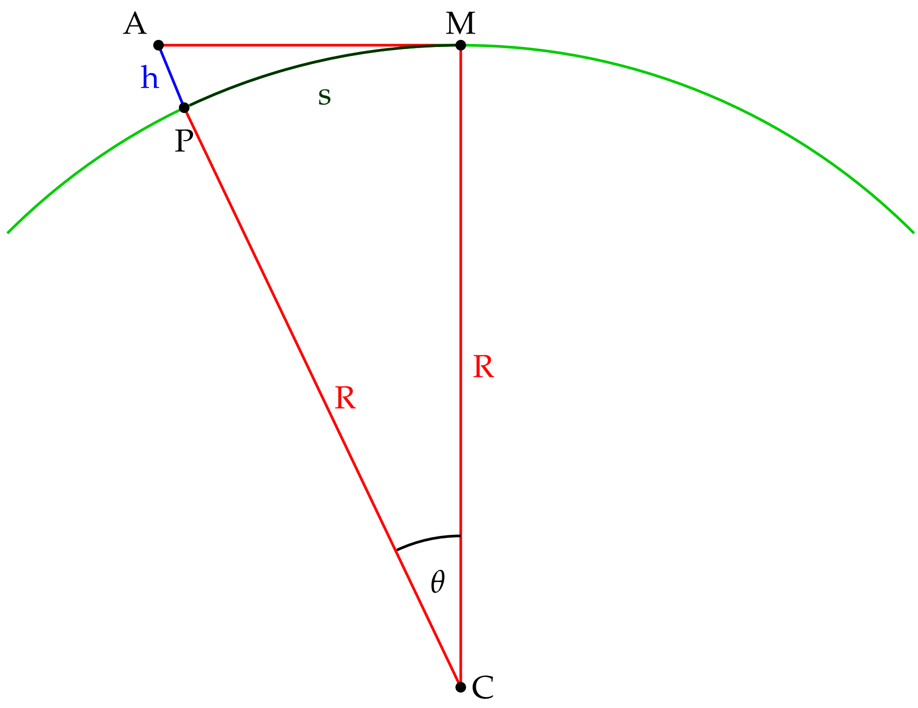
First, assume that the Earth is spherical, and that it has a radius of R. I'm going to place my radar on the surface (at M), and assume that the radar cannot see below the horizon (the line A to M). Assume that an aircraft (A) is at a distance s (which is measured around the curved Earth) from the radar, and that it is at a height h above the surface of the Earth (at P). We can call the radar-earth center-aircraft angle θ. See Figure 1.



Now from the definition of a radian, if the angle θ is measured in radians, then

$$s = R\theta$$

and so we could write

$$\theta = \frac{s}{R}$$

Now have a look at triangle AMC. Using SOH-CAH-TOA we can write

$$R = (R+h)\cos(\theta) = (R+h)\cos\left(\frac{s}{R}\right)$$

Multiply out the brackets:

$$R = R\cos\left(\frac{s}{R}\right) + h\cos\left(\frac{s}{R}\right)$$

Subtract $R\cos\left(\frac{s}{R}\right)$ from both sides:

$$R - R\cos\left(\frac{s}{R}\right) = h\cos\left(\frac{s}{R}\right)$$

and divide both sides by $\cos\left(\frac{s}{R}\right)$ we get

$$\frac{R}{\cos\left(\frac{s}{R}\right)} - R = h$$

This can be written as

$$h = R\left(\sec\left(\frac{s}{R}\right) - 1\right) \tag{1}$$

using the definition of $\sec(x)$, and factorising the R outside the brackets. It is possible to obtain an infinite series expansion of $\sec(x)$ as a polynomial in x. That's the same kind of thing that you get when you do binomial expansions. It turns out that the series expansion for $\sec(x)$ is

$$\sec(x) = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + ...$$

Now if x is very small, then $x^4 << x^2$, and $x^6 << x^4$, etc. In that case,

$$\sec(x) \approx 1 + \frac{1}{2}x^2$$

In our radar situation, radars only see out to a range of about 250 miles (s), whereas the radius of the Earth (R) is about 4000 miles. So $s << R$, and we can approximate $\sec(\frac{s}{R})$ by

$$\sec\left(\frac{s}{R}\right) \approx 1 + \frac{1}{2}\left(\frac{s}{R}\right)^2$$

So,

$$h = R\left(1 + \frac{1}{2}\left(\frac{s}{R}\right)^2 - 1\right)$$

or

$$h = R\left(\frac{1}{2}\left(\frac{s}{R}\right)^2\right)$$

or

$$h = \left(\frac{1}{2R}\right)s^2 \tag{2}$$

Whew!

To see how good the approximation is, here's a table showing the values of (1) and (2) for different values of s from 0 to 250 miles, using $R = 4000$ miles.

Table 1: Comparing exact and approximate formulae

s	$R\left(\sec\left(\frac{s}{R}\right) - 1\right)$	$\left(\frac{1}{2R}\right)s^2$
50	0.31252	0.31250
100	1.25033	1.25000
150	2.81415	2.81250
200	5.00521	5.00000
250	7.82524	7.81250

The largest error being for the case of $s = 250$ miles. The error is approximately 77 feet in height! I'm happy with that!