Steve Smith Tuition: Physics Notes

Physics Key Skills : Vectors

Contents

Notes

None.

Document History

Date	Version	Comments
25th October 2012	0.1	Initial creation of the document.

1 Prerequisites

Here is a summary of the ideas explained elsewhere that this document uses.

1.1 A Bit of Trig

Here's my replacement for SOH CAH TOA, which is fully explained in Ref [Smith(2012b)]. You should go away and read that ***now***!

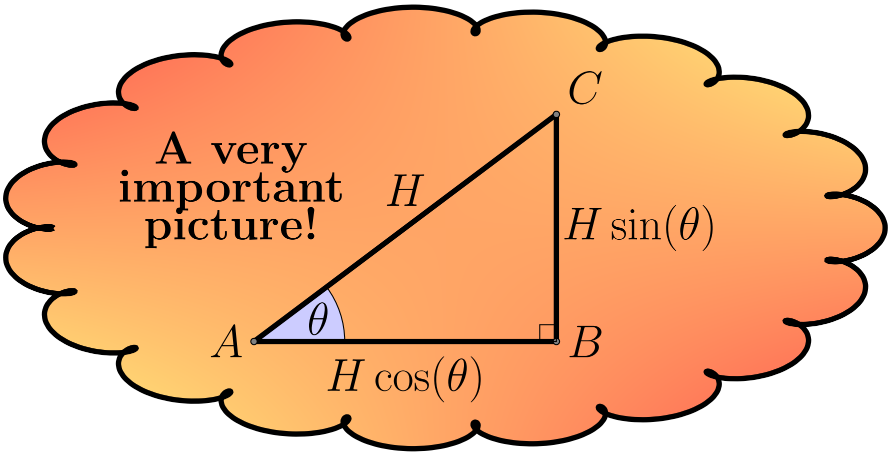


Figure 1: A Picture to Remember!

Which says that: if the length of the hypotenuse (the long side) of a right-angled triangle is H, and one of the angles is θ, then you can simply write down that the length of the side opposite the angle θ will be $H\sin(\theta)$, and the length of the side next to the angle θ will be $H\cos(\theta)$.

1.2 A Bit More Trig

Here is the definition of $\tan(\theta)$, which is fully explained in Ref [Smith(2012b)]. You should go away and read that ***now***!

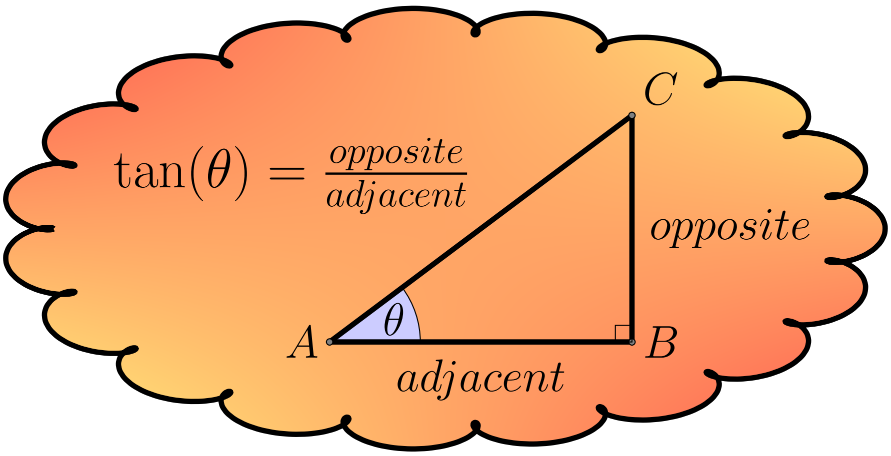


Figure 2: The Definition of $\tan(\theta)$

1.3 Pythagoras' Theorem

Here is Pythagoras' Theorem, which is fully explained in Ref [Smith(2012b)]. You should go away and read that ***now***!

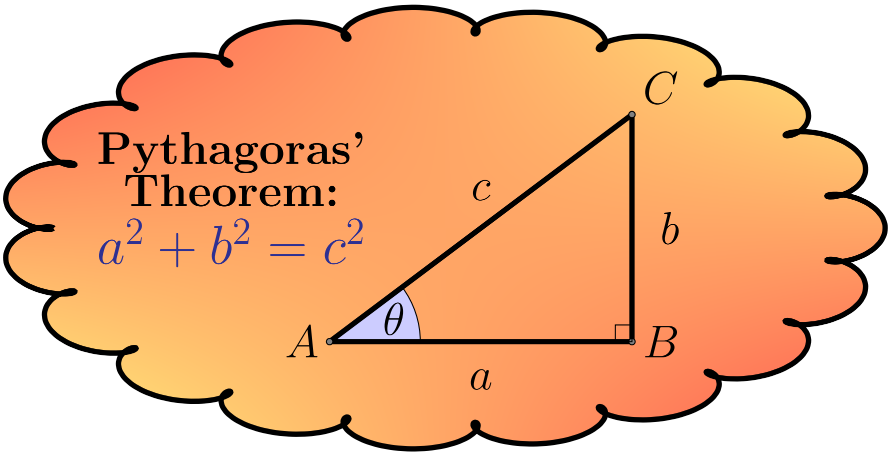


Figure 3: Pythagoras' Theorem

As seen in Figure 3, if you have a right-angled triangle, and c is the length of the hypotenuse (the long side), and a and b are the lengths of the other two sides, then

$$a^2 + b^2 = c^2$$

2 What Is A Vector?

Now then. You've heard about vectors, but what on Earth are they?

Before we talk about vectors, it's a good idea to have a quick recap of ordinary numbers. So: most of the quantities that you have come across so far in your life only have a size. Here are a few examples:

- A temperature (e.g. 273 K);
- The volume of the room you are in (e.g. 24 m^3);
- The mass of an apple (approximately 0.1 kg);
- The distance to the moon (about 3.84×10^8 m);
- My age (21, of course);
- The speed of my car as I travel up the M1 (about 22 ms^{-1}).

In all of these examples, we only need one number to describe them.

Now the last item in that list could contain more information: as well as the size of my speed (22 ms^{-1}), you might be interested to know which direction I'm going in. So, if I said that I was going 22 ms^{-1} *North*, then I am containing both a direction and a size in my information. This kind of size-and-direction thing is represented by a *vector*. Here are some more vectors:

- Tesco is 150 m *South West* from my house;
- The weight of an apple is about 1 N, *vertically down*;
- The velocity of my car is 25 ms^{-1} *on a bearing of* 125°.

Notice that I have used the word *velocity* rather than speed for the last example. That's because in Physics, *speed* refers to an ordinary number, which just has a size[1], whereas a *velocity* has a size but a direction as well. So 30 ms^{-1} is a *speed*; 30 ms^{-1} *South* is a *vector*.

And so, put simply:

a vector is a quantity
that has both a size,
and a direction

[1]This is actually called a *scalar* quantity.

3 How Vectors Add

So now you know what a vector is, how do we go about trying to draw one? And how might we manipulate such things? For example, if a vector is just a kind of two-dimensional number (because it comprises of two bits: the size, and the direction), could we add two of them together?

Well, yes you can! And knowing how to add two vectors together is a very useful skill in Physics. It may seem odd that you can add vectors together. But since a vector is just a kind of two-dimensional number, you can add two of them together. And in fact, you can do lots of things with vectors that you can do with ordinary numbers. You can add vectors. You can subtract one vector from another. You can multiply vectors[2]. The only thing that you can't do with vectors that you can do with ordinary numbers is to divide by a vector. [By the way, if you want to find out how you can do some of this stuff, check out Appendix A.]

Have a look at Figure 4.

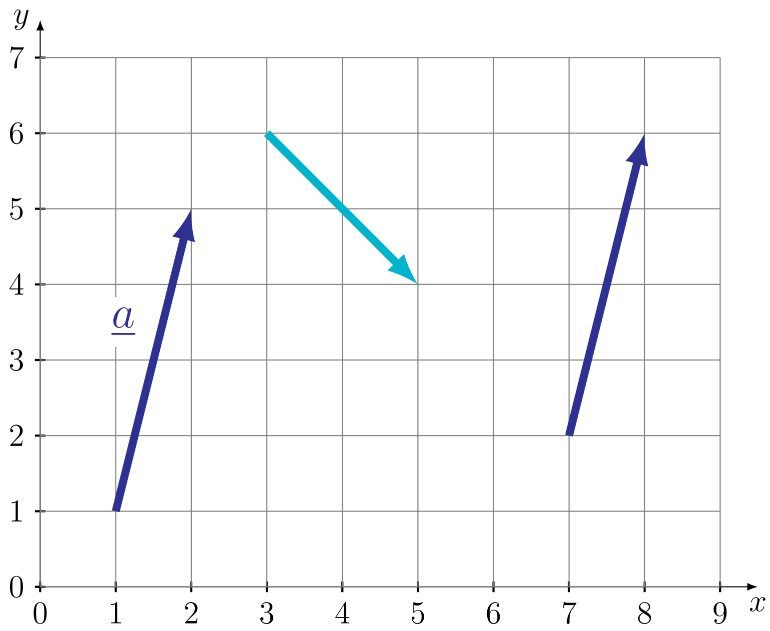


Figure 4: Some Vectors

In Figure 4 we have three vectors. This is the normal way that we visualise a vector: we draw a line with an arrow on the end. That's because a vector is a number that has both a size and a direction. We can summarise both in a simple line with an arrow on the end: the length of the line represents the size of the vector; the arrow shows you the direction.

OK. So that's how we can draw a vector. But what about algebra? Could we write equations with vectors in? After all, we would need to set up an equation to add two vectors together, wouldn't we?

In order to do that, we would need a way of writing a vector down algebraically. There are many ways of doing this, but the one I want to use here is one that uses the (x, y) coordinate grid that I hope by now you are pretty familiar with.

Have a look at the vector on the extreme left of Figure 4. That's the one marked $\underline{a}$. That vector starts at $(1, 1)$ and ends at $(2, 5)$. So we could describe it like that: it starts at $(1, 1)$ and ends at $(2, 5)$. But that's a bit of a mouthful. Is there a more convenient, more efficient notation we could use? Well, how about this:

[2]In fact, there are *two* ways of doing this!!

$$\underset{\sim}{a} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

Can you see that this is quite neat: the vector goes 1 to the right, and 4 up. And really, that's all we need to know about it. That describes perfectly both how far to go (the size of the vector), and which way to go (the direction). So in this notation, the number on the top of the column denotes how far *right* the vector is going (an x-thing), and the number on the bottom denotes how far the vector is going *up* (a y-thing).

Notice, by the way, that the other **blue** vector in Figure 4 also goes 1 to the right and 4 up. So, even though it is drawn in a different place, because it has the same length as $\underset{\sim}{a}$, and it is going in the same direction as $\underset{\sim}{a}$, that second blue vector is also $\underset{\sim}{a}$!

Right. Now how about the middle vector in Figure 4. How could we describe that algebraically? Can you see that if we use the same idea that we have just used for $\underset{\sim}{a}$, then we could write $\underset{\sim}{b}$ as:

$$\underset{\sim}{b} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

See Figure 5. That's because $\underset{\sim}{b}$ goes 2 to the right, but there's a -2 on the bottom because the vector goes 2 *down* (and not up).

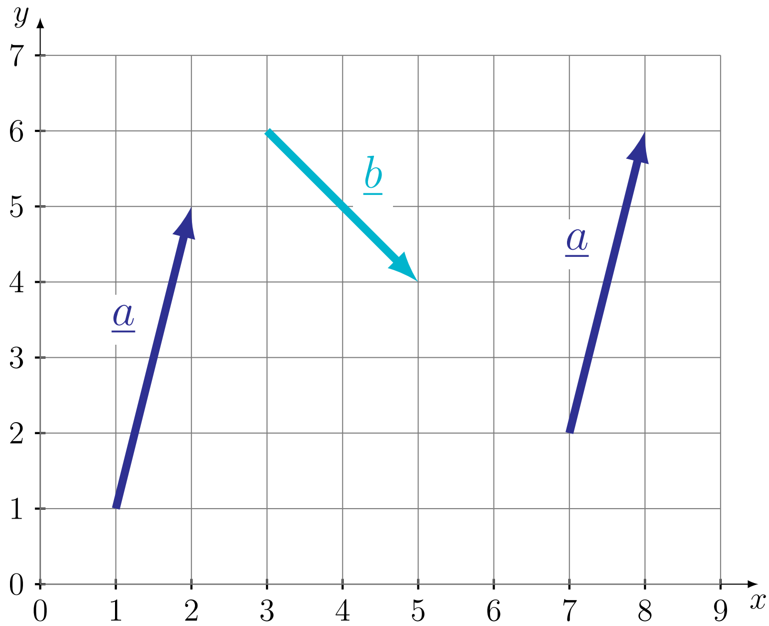


Figure 5: The Vectors With Names!

Cool! But how does this get us closer to being able to add two vectors together? Well, if you now forget about the pictures for a moment, and concentrate on the algebra, then we can now actually add $\underset{\sim}{a}$ and $\underset{\sim}{b}$ together! Can you guess how we could do this:

$$\underset{\sim}{a} + \underset{\sim}{b} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} + \begin{pmatrix} 2 \\ -2 \end{pmatrix} = ??$$

Well, hopefully you have spotted that we could add $\underset{\sim}{a}$ and $\underset{\sim}{b}$ together by doing this:

$$\underset{\sim}{a} + \underset{\sim}{b} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} + \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

We get the 3 on the top by just adding the tops (the x-bits) of our vectors together; we get the 2 on the bottom by adding the bottoms (the y-bits). OK, so what does the vector $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ look like? Can we include it in our picture? Yes we can! And for reasons that I will explain later, I'm going to start this vector at the start of $\underline{a}$. See Figure 6.

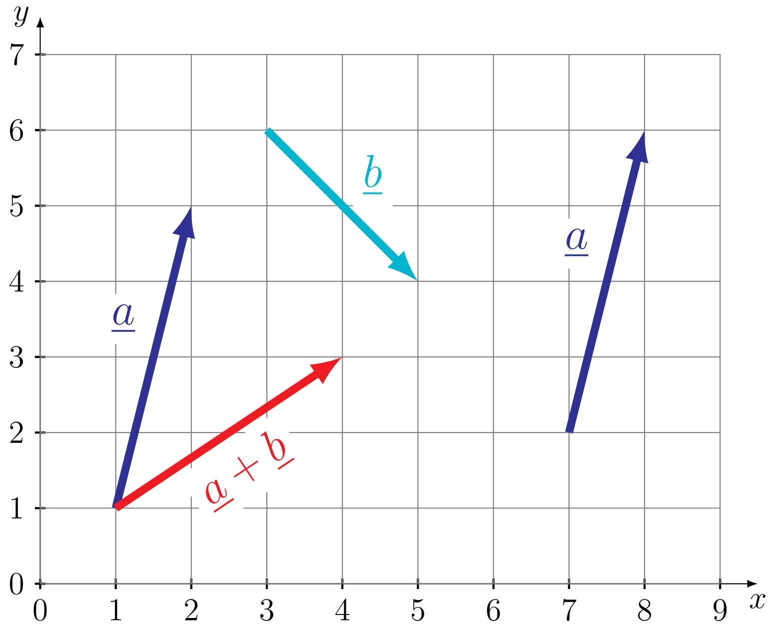


Figure 6: Including the Vector $\underline{a} + \underline{b}$

Now to completely visualise how we add two vectors $\underline{a}$ and $\underline{b}$ together, it would be really nice if we could have a little picture that includes $\underline{a}$ and $\underline{b}$ and also $\underline{a} + \underline{b}$. Have another look at Figure 6. Can you see how we might do that? Well, because it is not important at all where we draw the vector $\underline{b}$, so long as it goes 2 to the right and 2 down, how about drawing it like I do in Figure 7:

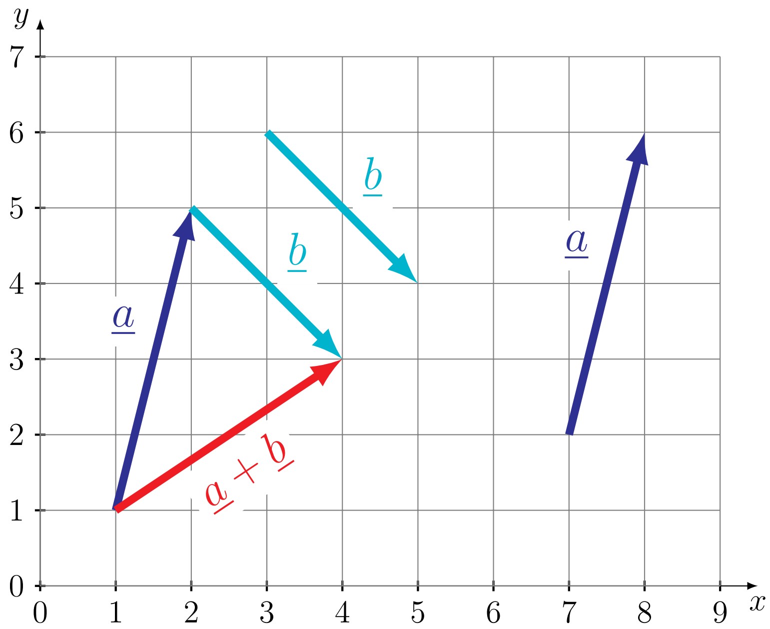


Figure 7: How To Add $\underline{a}$ and $\underline{b}$

All I've done is slide the original $\underline{b}$ down a bit and left a bit to fit into the gap from Figure 6. And now we have a neat way of visualising how to add two vectors together. Here's the picture again, without all the clutter:

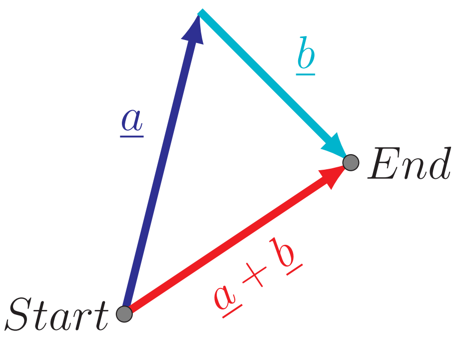


Figure 8: Adding Two Vectors

In order to add two vectors together, $\underline{a} + \underline{b}$, say, what you have to do is to start at the beginning of $\underline{a}$, travel to the end of $\underline{a}$; move the other vector ($\underline{b}$) so that it starts at the end of $\underline{a}$, then travel along that. Now draw a vector from where you started to where you end up, and that is your vector $\underline{a} + \underline{b}$.

And that's how vectors add!

Here's another example:

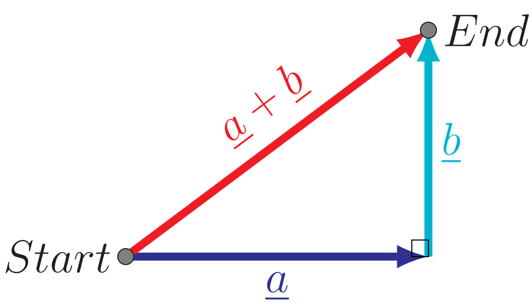


Figure 9: Another Example of Adding Two Vectors

And another:

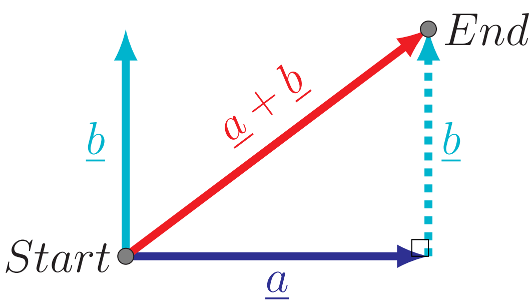


Figure 10: Yet Another Example of Adding Two Vectors

This is quite an interesting one, bearing in mind what's coming up in Section 4. In Figure 10 I've drawn both $\underline{a}$ and $\underline{b}$ starting at the same point. But can you see that I can still draw $\underline{a} + \underline{b}$ in the same place as Figure 9, because I could slide $\underline{b}$ over to the end of $\underline{a}$ (shown with a dashed line) and draw my little triangle of $\underline{a}$, $\underline{b}$ and $\underline{a} + \underline{b}$ that way.

4 Finding the Components of a Vector

Well, before we can understand how to find components of vectors, we have to know what components of vectors are! So, what are vector components?

Put simply, the components of a vector $\underline{R}$ are any vectors that add to make $\underline{R}$.

But to make components useful, here's a working definition: components of a vector $\underline{R}$ are any *two* vectors *at right-angles* to each other, that add to make $\underline{R}$.

Have a look at Figure 11 for a few examples. In each case, $\underline{R} = \underline{a} + \underline{b}$.

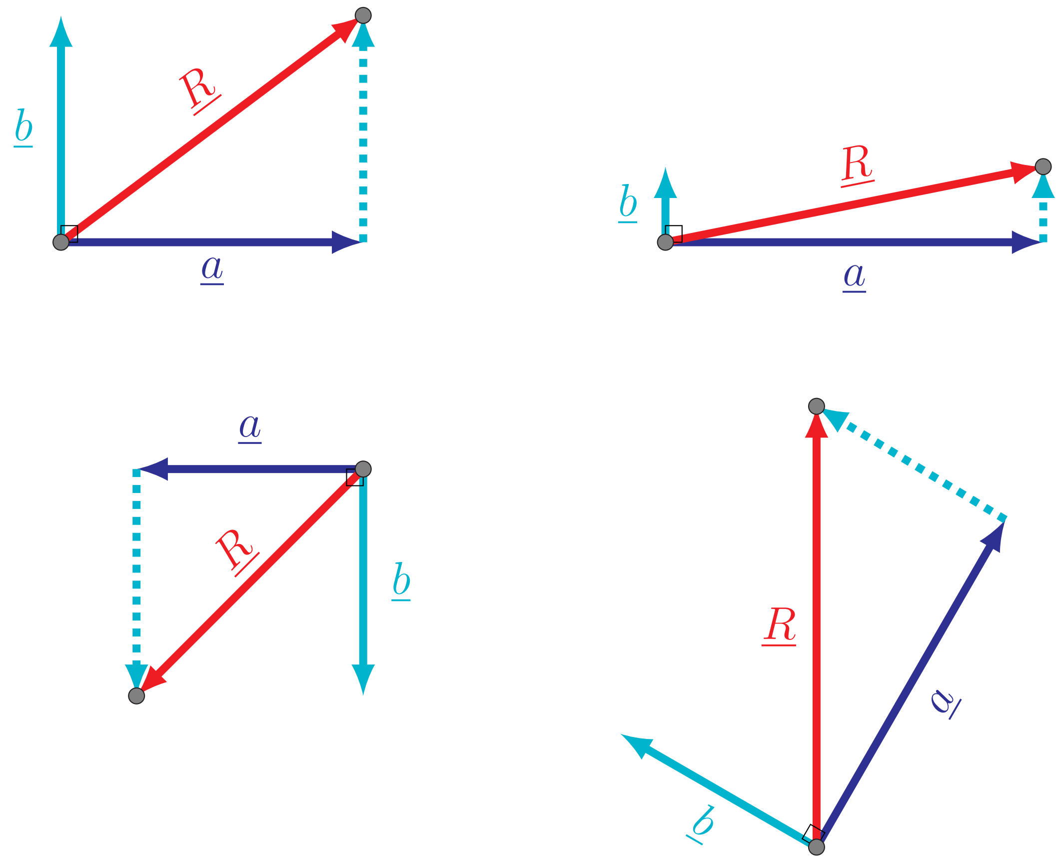


Figure 11: Examples of *Useful* Vector Components

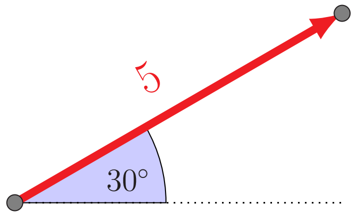
Notice how I've drawn the vectors $\underline{a}$ and $\underline{b}$ in each case. I've drawn them all coming out of the point that's the start of our actual vector $\underline{R}$. The reason for this is that in Physics, vectors are used most often to represent forces. Now in the above diagrams, if $\underline{R}$ is a force, then $\underline{a}$ and $\underline{b}$ will be forces as well. And if $\underline{R} = \underline{a} + \underline{b}$, then the forces $\underline{a}$ and $\underline{b}$ acting together *are equivalent to* the vector $\underline{R}$. So we could replace $\underline{R}$ by $\underline{a}$ and $\underline{b}$ and the object would still feel the same effects of the forces. Now if I was the object, I would feel these forces pulling me from where I am toward somewhere else. That's why I always draw vector components starting from the object.

So that's why I have slid the dotted component over to start at the object in each case. And from now on in this document, that's how I will draw my components. They're both going to start at the beginning of the vector they will be replacing.

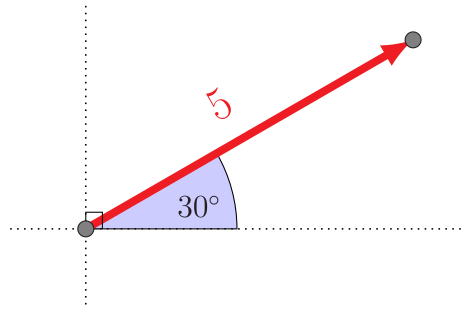
OK. So here's the problem: if you start with a vector, $\underline{R}$, that you know, then how do you find the components $\underline{a}$ and $\underline{b}$? Let's work through an example to show you how you do it.

4.1 Finding Components : Example 1

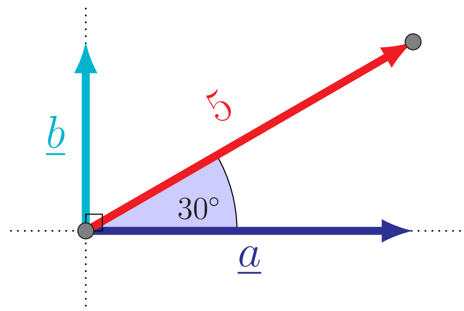
Let's say you started with this vector:



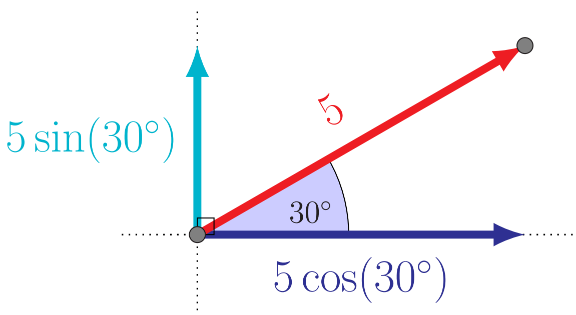
What we know is: the length of the vector is 5 and its direction is at 30° to the horizontal. So where do we start? Well, the first thing we need to do is to pick two directions that our components will lie in. And the only important thing about those directions is that they have to be at right-angles to each other. In this case, the directions I'm going to pick are *vertical* and *horizontal*:



So now we can draw the components of this vector in. They have to lie along my chosen directions, and they have to add to make the original vector:



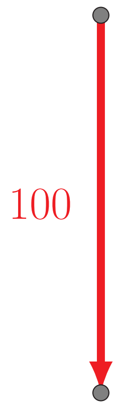
Right. Nearly there! So all we have left to do is to find the lengths of a and b. And how do we do that? We use the idea from Ref [Smith(2012b)] (see Section 1.1):



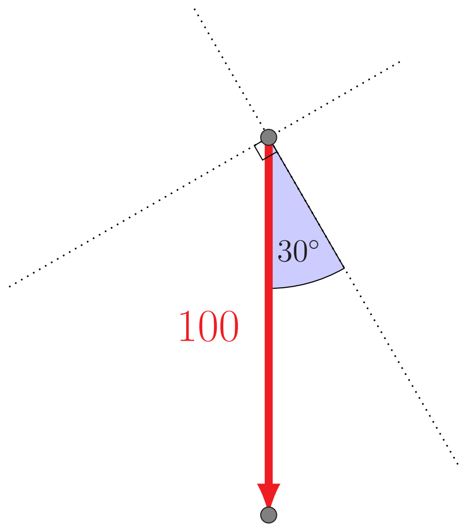
And we've done! That's how you find the components of a vector!

4.2 Finding Components : Example 2

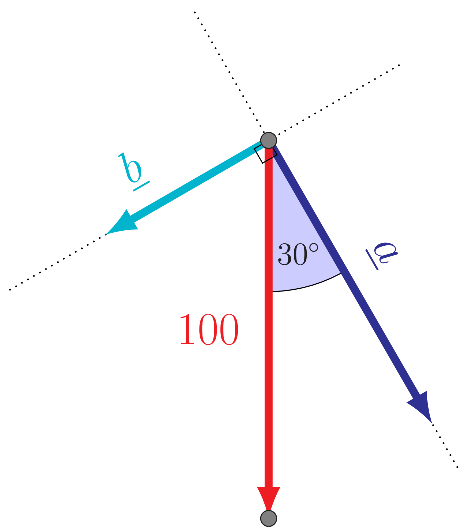
This time we're going to start with this vector:



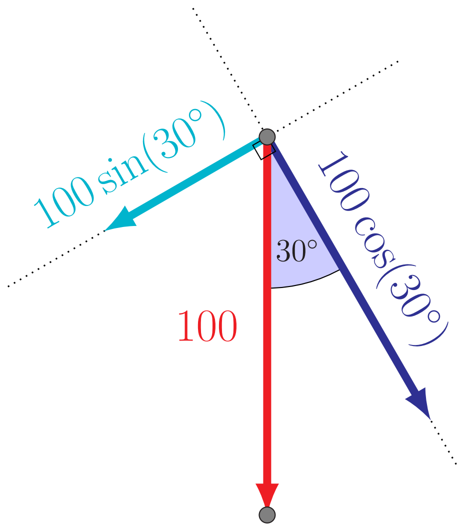
What we know is: the length of the vector is 100 and its direction is vertically down. Now this time I'm asking you to find components in these directions:



Well I'm hoping by now that you can see how we can draw the components of this vector in. They have to lie along my chosen directions, and they have to add to make the original vector:



Right. Nearly there! So all we have left to do is to find the lengths of a̲ and b̲. And how do we do that? We use the idea from Ref [Smith(2012b)] (see Section 1.1):



And we've done! We've found the components of another vector!

5 Why Components Are Useful

So: what's the point of all this? Why would we need to split a vector up into components? Wasn't the original vector good enough? Haven't we made things worse by having two vectors now when before we only had one?

Let me answer these questions by a couple of examples.

5.1 Example 1: An Object in Equilibrium

Let's say we had an object, and it had three forces on it, as shown in Figure 12. The question is: find S and T.

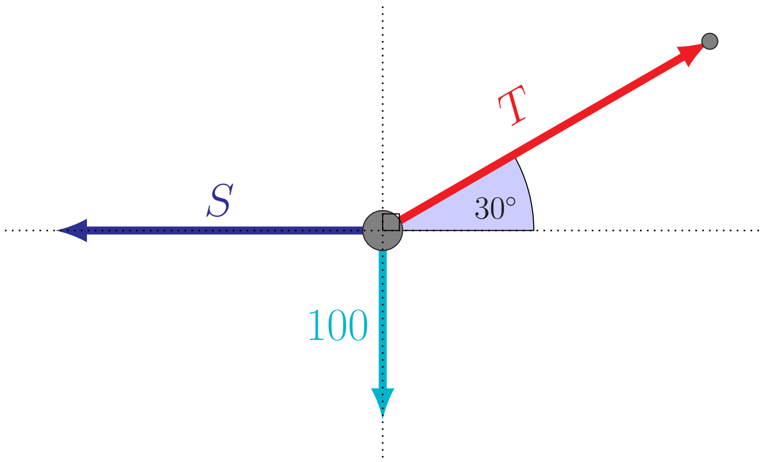
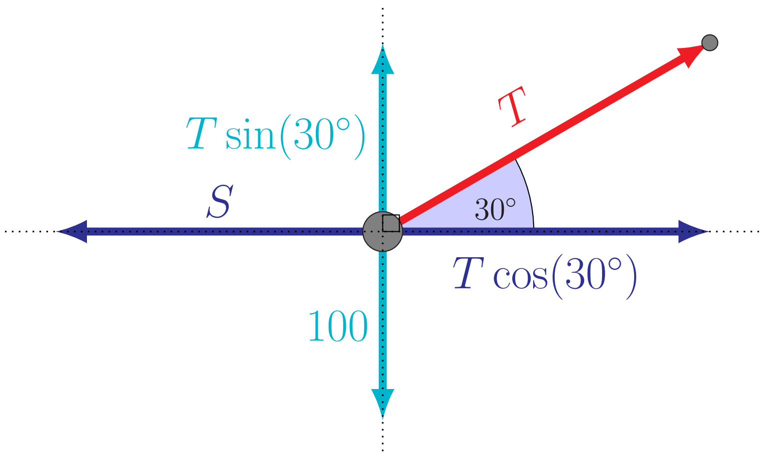


Figure 12: How Components Can Be Useful : Example 1

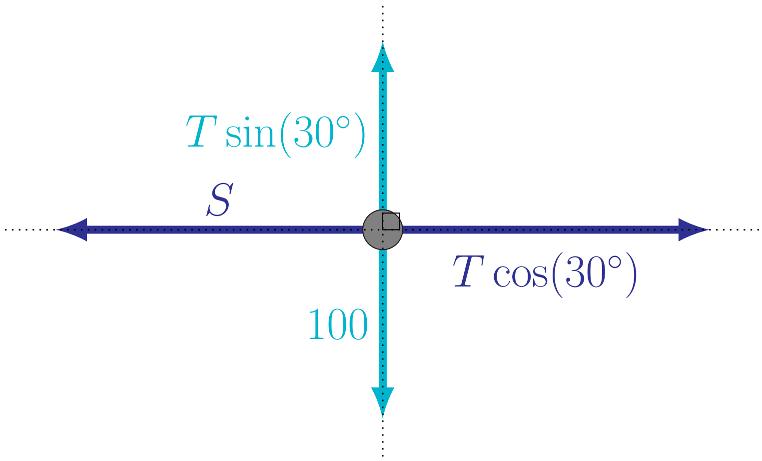
We've also been told that the object is in *equilibrium*. Now what does *equilibrium* mean? Put simply, equilibrium means that the object has no net force on it in any direction[3].

So how can we use that fact? Right - this is where the components of a vector come in handy. If we find the components of T in the two directions indicated:



then, because of the way vectors add, we can replace the vector T by it's components:

[3]Another way of putting it would be that the object is not accelerating. By Newton's Second Law, $F = ma$, this means that if there is no a, then there is no F.



And how does this help? Well, if you think about it, if you had forces pulling this object left and right (like the S and $T\cos(30°)$ forces here), no matter how big those forces were, they would never be able to make the object go up or down. Similarly, if you had forces pulling the object up or down (like the 100 and $T\sin(30°)$ forces here), then no matter how big they are, they can't make the object go left or right.

Big idea: as we have split all the forces that act on this object up into two perpendicular directions, *we can treat the forces in each direction separately.*

we can analyse the forces on an object in two perpendicular directions *separately*

Consequently, just looking at the forces in the vertical direction, since the object is in equilibrium, these forces must balance:

$$T\sin(30°) = 100 \tag{1}$$

and just looking at the forces in the horizontal direction, since the object is in equilibrium these forces must balance too:

$$S = T\cos(30°) \tag{2}$$

And in order to solve our problem, we need to solve equations (1) and (2). To do that, all we have to do is to use equation (1) to find T (how?), and then plug that value of T into (2) to find S.

5.2 Example 2: An Object on an Inclined Plane

A very popular question type in Physics and Mechanics exams in Maths is the old "Object on an inclined plane". Have a look at Figure 13 to get the idea. We have an object, and it is on a slope inclined at 30° degrees to the horizontal. Again, we are told that the object is in equilibrium.

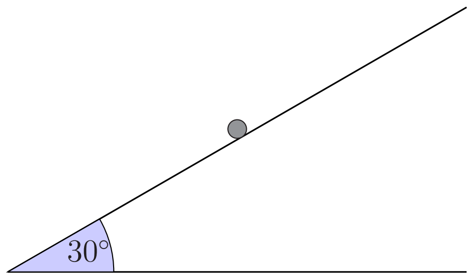


Figure 13: An Object on an Inclined Plane

Now when analysing situations like this, the first thing we have to do is to draw in all the *real* forces. So - what are they going to be?

Well, objects always have a weight, so we need a weight force. That's going to act vertically down from the object.

Then there's the normal reaction force. There's always one of those when an object is in contact with a surface. And the direction of this force is perpendicular (i.e. normal) to the surface.

Finally, there's friction. And which way would that act? Well, friction always acts to try and stop an object from moving. This object would try to slide down this slope, so friction here would try and prevent that by acting up the slope.

See Figure 14 showing the real forces on the object.

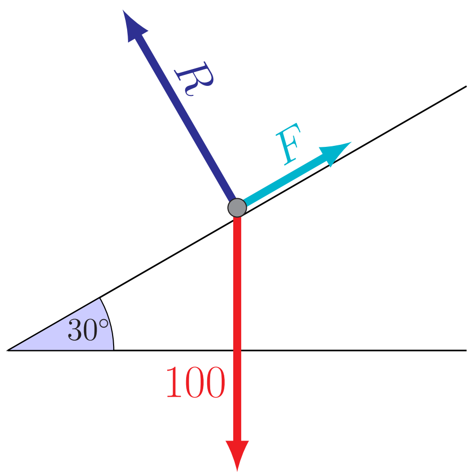


Figure 14: The Real Forces on the Object

Now we're going to use that idea of splitting up forces so that they all lie in perpendicular directions. That way we can treat the forces in each of those directions separately. But which directions should we choose for our perpendicular directions?

Usually the most obvious choice is vertically and horizontally. But sometimes, choosing the perpendicular directions other than vertically and horizontally makes more sense. And this is one of those situations. I'm going to choose my directions to be normal to the plane, and perpendicular to the plane. See Figure 15.

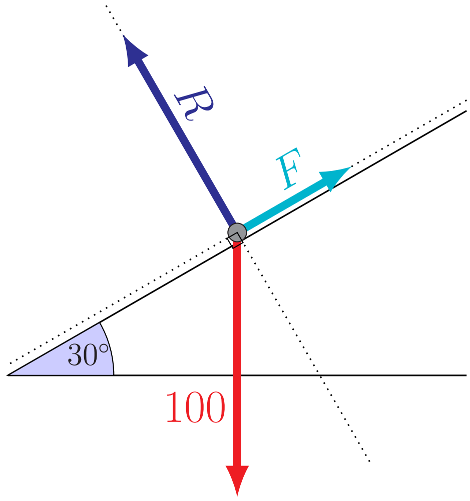


Figure 15: The Choice of Perpendicular Directions

Now why on Earth have I chosen the perpendicular directions to be those?

Well the idea is to put all of our vectors in the perpendicular directions. And if we choose along the plane and perpendicular to the plane then two of our three vectors are already in those directions. That means we don't have to find the components of those. This way, we only need to find the components of the weight force in those directions. So let's do that next. I hope you remember how to do this by now!

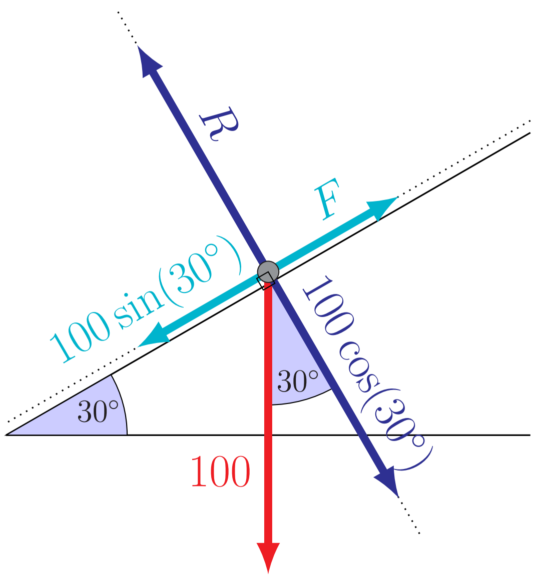


Figure 16: Finding the Components of the Weight

And remember that now we have the components of the weight in our chosen directions, then because of the way vectors add, we can replace the weight by the components.

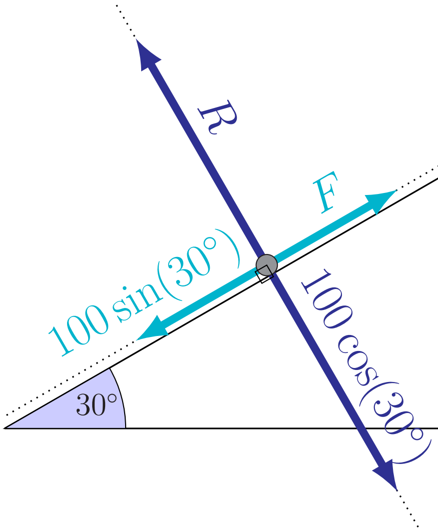


Figure 17: All Forces Fully Resolved

And we have now completed our diagram. All the forces in the problem have been resolved into ones that lie in two perpendicular directions.

Now let's get on with solving the problem. Remember the big idea? It's that we can treat the forces in the two directions separately. Now because the object is in equilibrium, then the forces in each direction must balance. So we have

$$100\cos(30°) = R \tag{3}$$

and

$$100\sin(30°) = F \tag{4}$$

and these equations are enough for us to find the forces R and F.

6 Finding the Resultant Vector from Components

Right, well this section is the exact opposite of Section 4. There we found the components from the proginal vector. Here, we know the components, and we are trying to find the resultant of them.

6.1 Example 1

Let's say that you had the components shown in Figure 18.

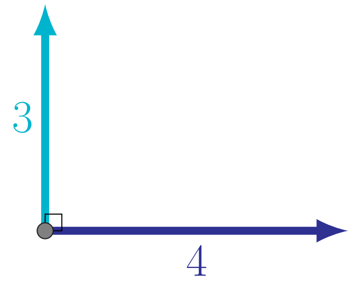
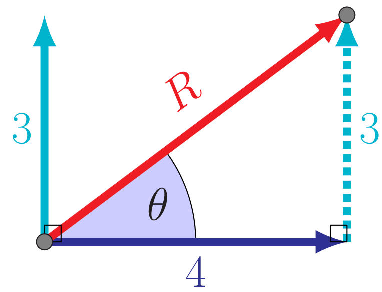
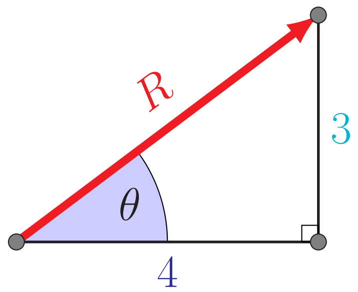


Figure 18: Finding the Resultant of Two Components

What we need to do here is the find the vector that these two components add together to make. OK, so knowing how to add components together (slide the second vector along so that it starts at the end of the first, etc etc) we can draw a picture of the resultant vector R:



and now, if we get rid of all the clutter from this picture:



the problem comes down to this: can we find from the above triangle the length R of the resultant vector and the angle θ that it makes with the horizontal?

Well, if you have read and inwardly digested by trigonometry notes (Ref [Smith(2012b)]), then you will know how to do this. To find the length R of the hypotenuse (the longest side) of this triangle, you use Pythagoras' Theorem:

$$R^2 = 4^2 + 3^2$$

so that

$$R \;=\; \sqrt{4^2 \;+\; 3^2} \;=\; 5$$

And to find the angle θ, you just use the definition of $\tan(\theta)$, which boils down to

$$\tan(\theta) \;=\; \frac{3}{4}$$

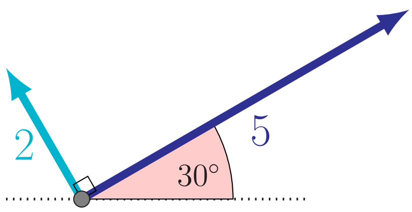
so that (using inverse functions: see Ref [Smith(2012a)])

$$\theta \;=\; \tan^{-1}\left(\frac{3}{4}\right) \;=\; 36.9^\circ \;(to\ 3\ sf)$$

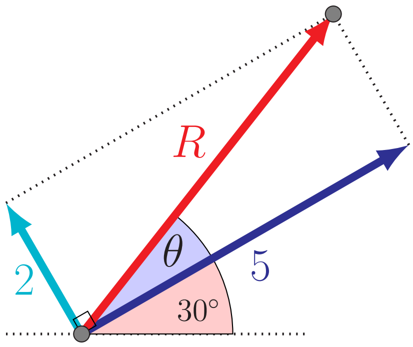
And this is how you find the resultant of a vector from it's components. You will use Pythagoras to find the length of the resultant, and the definition of $\tan(\theta)$ to find the appropriate angle.

6.2 Example 2

Here's a final example. Find the resultant of these components. When you find the direction of the resultant, I want it to be relative to the horizontal.



The first thing to do is to draw in the resultant vector:



Then, as before,

$$R \;=\; \sqrt{5^2 \;+\; 2^2} \;=\; 5.39 \;(to\ 3\ sf)$$

and

$$\theta \;=\; \tan^{-1}\left(\frac{2}{5}\right) \;=\; 21.8^\circ \;(to\ 3\ sf)$$

So in order to find the direction of R relative to the horizontal, we simply need to add 30° to θ:

$$direction\ of\ R \;=\; 21.8^\circ \;+\; 30^\circ \;=\; 51.8^\circ \;above\ the\ horizontal \;(to\ 3\ sf)$$

A More Vector Stuff...

A.1 Negative Vectors

You know how to draw a vector $\underline{a}$ now. But what do you think the vector -$\underline{a}$ would look like? Well, going back to our beloved $\underline{a}$ from Section 3:

$$\underline{a} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

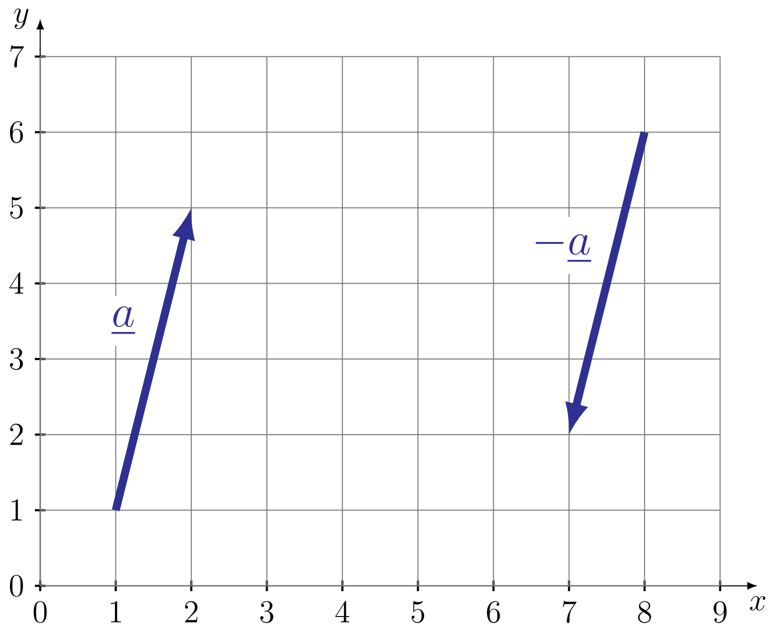
then -$\underline{a}$ would be

$$-\underline{a} = -\begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

wouldn't it? Or, in other words,

$$-\underline{a} = -\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ -4 \end{pmatrix}$$

Is that resonable? Let's see what that vector would look like:



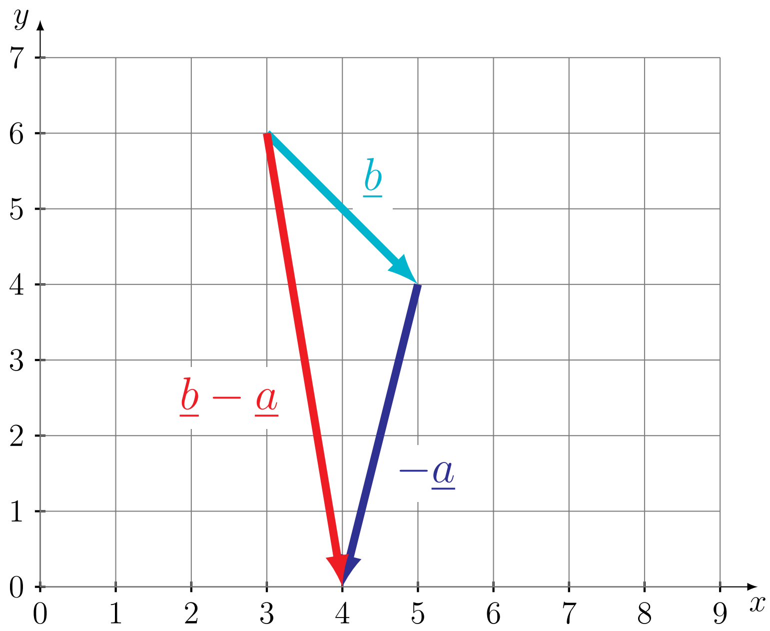
This is just the vector $\underline{a}$ but going the opposite way! Cool! So that's how you can interpret a negative vector: -$\underline{a}$ just a vector that is the same length as $\underline{a}$, but is going in the opposite direction.

A.2 Subtracting Vectors

Now we know how to interpret the negative of a vector, this gives us a neat way of subtracting vectors. That's because you could write $\underline{b}$ - $\underline{a}$ as

$$\underline{b} - \underline{a} = \underline{b} + (-\underline{a})$$

which means that you can subtract a vector by just adding it's negative vector. Here's an example:



This example shows that if you add -$\underline{a}$ to $\underline{b}$ you get $\underline{b}$ - $\underline{a}$.

A.3 Multiplying Vectors By a Scalar

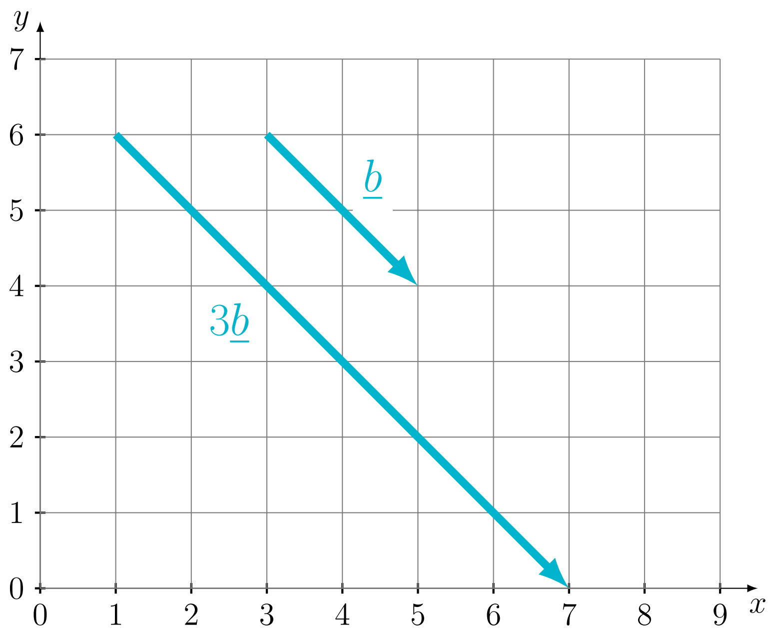
OK, so if you know what a vector $\underline{b}$ is, what do you think $3\underline{b}$ means? Let's have a look. If

$$\underline{b} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

then $3\underline{b}$ would be:

$$3\underline{b} = 3\begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \end{pmatrix}$$

Can you see that? Now let's see what that vector would look like:



So $3\underline{b}$ is just a vector that's going in the same direction as $\underline{b}$, it's just 3 times as long.

References

[Smith(2012a)] **Smith, S.** (2012a). Maths Key Skills : Inverse Functions. An explanation of the idea of Inverse Functions, and how they are used to solve equations.

[Smith(2012b)] **Smith, S.** (2012b). Maths Key Skills : Trigonometry. An explanation of the five key skills in trigonometry.