Steve Smith Tuition: Physics Notes

Obtaining Schroedinger's Equation

Contents

Prerequisites

None.

Notes

None.

Document History

Date	Version	Comments
18th April 2013	1.0	Initial creation of the document.

1 Introduction

Schroedinger's Equation can't be derived. But we can *obtain* it from considering the energy of a particle. If we let E be the total energy of the particle, E_k being its kinetic energy, and U being its potential energy, then

$$E = E_k + U$$

since for any particle, its total energy is given by the sum of its kinetic and potential energies.

2 Useful Results from Quantum Mechanics

2.1 de Broglie's Formula

At the beginning of the last century, quantum mechanics was turning physics upside down. From experiments such as the photoelectric effect it was beginning to be understood that waves can sometimes behave like particles. Might not particles sometimes behave like waves, then?

In his PhD thesis of 1924, de Broglie postulated a formula that might connect wave-like stuff and particle-like stuff. He started with the famous

$$E = mc^2$$

formula, which Einstein came up with in 1905, which is to do with the equivalence of energy and the mass of a *particle*, and then combined it with the famous

$$E = hf \tag{1}$$

formula, which Einstein came up with in 1905 (it was a good year for him!), which is the idea that light (which until 1905 as thought of as a wave) is made of photons (i.e. particles!), each of which has an energy given by hf, where h is a constant, now known as Planck's constant.

Now de Broglie's idea was that if particles can sometimes behave like waves, and vice-versa, then maybe these energies are actually the same, so that we can write E, the energy of the 'thing' (whatever it is!) as:

$$E = hf = mc^2$$

And maybe this works for particles that don't travel at the speed of light too:

$$hf = mv^2$$

And because of the wave equation $v = f\lambda$, we can write

$$\frac{hv}{\lambda} = mv^2$$

which simplifies to

$$\frac{h}{\lambda} = mv = p \tag{2}$$

So here we are: equation (2) proposes that we can find the wavelength of a particle, or the momentum of a wave. Cool! The left-hand side of (2) is definitely wave-like stuff: you can't get more wavey than wavelength! And the right-hand side of (2) is very definitely particle-like. Only particles have mass as well as velocity.

2.2 The Form of the Wave Function

Now any (one-dimensional) progressive wave, that is any wave moving with time t and position x, has an equation for its amplitude of

$$\Psi = e^{i(kx-\omega t)} \tag{3}$$

assuming that the amplitude is 1 when $x = 0$ and $t = 0$. If the amplitude is A, say, instead, then the following results will all apply, but the A cancels out on either side of the equations. So I'm going to ignore it for now.

And of course because of the idea that waves and particles seem to be two facets of the same kind of thing, why not try and find a wave equation that describes particles...

2.3 Position Derivatives of the Wave Function

There are now several results that we will need later. First, differentiating (3) with respect to x gives

$$\frac{\partial \Psi}{\partial x} = ike^{i(kx-\omega t)} = ik\Psi$$

Differentiating again,

$$\frac{\partial^2 \Psi}{\partial x^2} = (ik)^2 e^{i(kx-\omega t)} = -k^2\Psi \tag{4}$$

Now, since k is defined to be $k = \frac{2\pi}{\lambda}$, and $p = \frac{h}{\lambda}$ from (2), then

$$\frac{1}{\lambda} = \frac{k}{2\pi} = \frac{p}{h}$$

and so

$$k = \frac{2\pi p}{h} = \frac{p}{\hbar}$$

So equation (4) becomes:

$$\frac{\partial^2 \Psi}{\partial x^2} = -\frac{p^2}{\hbar^2}\Psi$$

or

$$-\hbar^2 \frac{\partial^2 \Psi}{\partial x^2} = p^2\Psi$$

or indeed,

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} = \frac{p^2}{2m}\Psi \tag{5}$$

2.4 Time Derivatives of the Wave Function

The only differential of (3) with respect to t that we need is the first:

$$\frac{\partial \Psi}{\partial t} = -i\omega e^{i(kx-\omega t)} = -i\omega\Psi$$

Now if we multiply both sides of this equation by $i\hbar$ we get

$$i\hbar\frac{\partial \Psi}{\partial t} = -i\hbar i\omega\Psi = \hbar\omega\Psi \tag{6}$$

which is another result we'll need later.

3 The Development of Schroedinger's Equation : The Time-Independent Equation

So, starting with

$$E = E_k + U$$

where we usually write the kinetic energy as

$$E = \frac{1}{2}mv^2 + U$$

then since momentum p is given by $p = mv$, then $p^2 = m^2v^2$, so $\frac{p^2}{m} = mv^2$, and so we can write the energy equation as

$$E = \frac{p^2}{2m} + U$$

Now if we multiply this through by the wavefunction, we get

$$E\Psi = \frac{p^2}{2m}\Psi + U\Psi$$

or, using (5) we can write this as

$$E\Psi = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + U\Psi \tag{7}$$

which is known as the "Time Independent Shroedinger Equation".

4 The Development of Schroedinger's Equation : The Time-Dependent Equation

If you take equation (1) and multiply and divide the right-hand side by 2π, you get

$$E = hf = \frac{h\ 2\pi f}{2\pi} = \hbar\omega$$

so then

$$E\Psi = \hbar\omega\Psi$$

Now using the result (6),

$$E\Psi = \hbar\omega\Psi = i\hbar\frac{\partial\Psi}{\partial t}$$

And substituting this into the Time-Independent Schroedinger Equation, (7), we get

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + U\Psi \tag{8}$$

which is known as the "Time-Dependent Schroedinger Equation".