Steve Smith Tuition: Maths Notes

Solving Trigonometrical Equations

Contents

Prerequisites

None.

Notes

None.

Document History

Date	Version	Comments
29 December 2017	1.0	Initial creation of the document.

1 Introduction

This document is intended to help you solve trigonometrical equations. Trigonometrical equations are tricky to solve, and when we try to tackle them there are lots of things that we have to consider.

For example, there's the algebra, usually involving trigonometrical identities. And there's the problem that trigonometrical functions are *periodic*, so we quite often get more solutions to a problem than the calculator gives us (which is only ever one). So we have to have some way of figuring out what the "missing" solutions would be.

And there are traps that you can fall into as well!!

So before we start to panic, let's start with an example. I'll take you through the problem, showing how I would solve it, but not really explaining why I'm doing the things that I do, but just showing you the overall scheme of things. Later I'll explain in more detail each of the steps involved.

1.1 Our First Problem

So: here's the first problem. Solve

$$5\sin(\theta) = 9\cos(\theta)$$

for values of θ in the range $0^\circ \leq \theta \leq 360^\circ$.

1.2 The Solution

First, we need to modify our equation a bit. Let's divide both sides by $\cos(\theta)$:

$$\frac{5\sin(\theta)}{\cos(\theta)} = 9$$

Now I'm going to divide both sides by 5. Remember that I'm not explaining *why* I'm doing this yet, just go with it for now. So, we get:

$$\frac{\sin(\theta)}{\cos(\theta)} = \frac{9}{5}$$

Now at this point I happen to know that

$$\frac{\sin(\theta)}{\cos(\theta)} \equiv \tan(\theta)$$

so that my equation becomes

$$\tan(\theta) = \tfrac{9}{5}$$

and this is the form of my original equation that I was trying to get.

At this point you might be thinking: "Ah - all I need to do now is to do that $\tan^{-1}(...)$ thing on the calculator, to give

$$\theta = \tan^{-1}\left(\tfrac{9}{5}\right) \approx 60.9^\circ$$

and the job's done, right?"

Well...not quite. The problem is that the $\tan(\theta)$ function (and this goes for the $\sin(\theta)$ and $\cos(\theta)$ functions too) is *periodic*, so there is actually more than one solution to this problem (even in the small range where we have to find solutions). Have a look at Figure 1 to see what I mean.

The blue lines in Figure 1 represent the graph of $y = \tan(\theta)$. The red line is the graph of $y = \frac{9}{5}$. So the solutions to the equation $\tan(\theta) = \frac{9}{5}$ will be the places where the red and blue lines cross. That's because at those places, the y-coordinate of the point will be $\tan(\theta)$ *and* $\frac{9}{5}$. And it seems that there are loads of these places!

But the calculator only gives you the solution $\theta = 60.9^\circ$. So how do you get the others?

Well, what you have to do is to use the symmetry of the graph to get the other solutions. In the case of Figure 1, the symmetry is that the $\tan(\theta)$ function repeats every 180°. So to get the other solutions, we add (or subtract) multiples of 180° to (or from) 60.9° to get the other solutions.

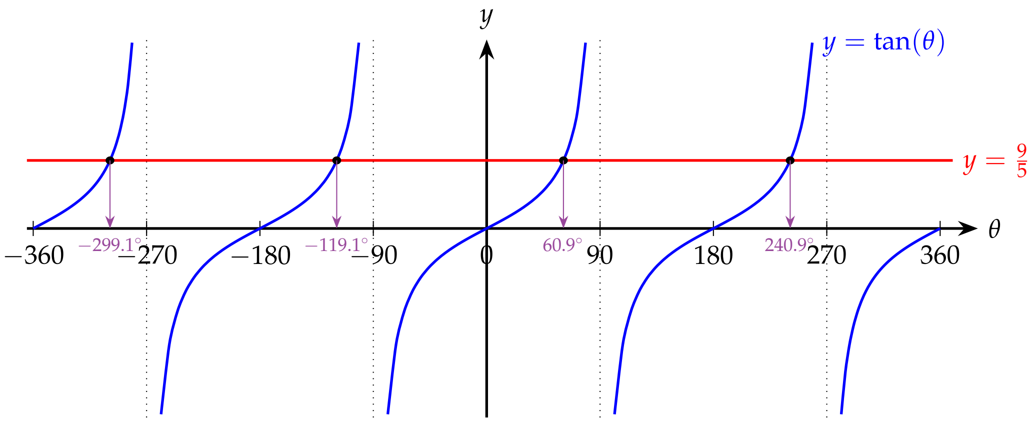


Figure 1: The graphs of $\tan(\theta)$ and $y = \frac{9}{5}$

1.3 The Cunning Plan!

OK, so this is how I think of solving trigonometrical equations. There are two parts to this cunning plan, and I've cunningly called them: "Part I" and "Part II". Cunning, eh?

1.3.1 Part I

Part I consists of massaging your original equation into one of the following forms:

$$\begin{aligned}\sin(\theta) &= \text{some number, or}\\ \cos(\theta) &= \text{some number, or}\\ \tan(\theta) &= \text{some number}\end{aligned}$$

Every trigonometrical equation to be solved MUST be converted into one of the above forms before you can solve it. Putting your equation into one of these forms is the job of Part I.

Sometimes, after you have worked your magic on the original equation, you end up with more than one of these. For example, if your equation to be solved turned out to be a quadratic, then we may get two of these things as the result of Part I.

1.3.2 Part II

Part II, very simply, solves the equation(s) you get as the result of Part I!

Let's say that after the end of Part I, you have

$$\cos(\theta) = \frac{1}{2}$$

What you need is to have some way for obtaining all the other solutions from the one that the calculator gives you. I only know of two such methods. These are

- drawing a graph, as in the above example, or
- using the *CAST* diagram (or it's equivalent!).

For the graph method you draw a graph of $y = \cos(\theta)$, and on the same axes, you draw a graph of $y = \frac{1}{2}$. You mark where your lines cross. Then you use the calculator to find one of the solutions, and use the symmetry of the cosine function to find the rest. That's my favourite method.

For the *CAST* diagram method, see Section 2.2.3.

2 The Cunning Plan : In Greater Detail

So, when we solve trigonometrical equations, we have a two-part process:

- Part I : here we have to end up with something along the lines of $\sin(x) = \frac{1}{2}$.
- Part II : find all the solutions of the equation you ended up with at the end of Part I. For that, you either draw the graph, or use the *CAST* scheme.

Of the two parts, Part I is usually the hard part. Part II is just following a recipe: we will be doing the same thing every time.

But Part I can often be very tricky. And there is no recipe. We will have to do something different every time, depending on circumstances.

2.1 Part I : What Options Do We Have?

So - if there's no recipe for doing Part I, what are the things that we could do? What are the skills that we will need to be able to do Part I? Well, here are some guidelines:

- we can use all the ideas of basic algebra, for example
  - adding, subtracting, multiplying or dividing algebraic fractions
  - *DOTS* - the *difference of two squares* comes up a lot
  - solving quadratics (particularly *hidden* quadratics)
- there are all the tigonometrical identities that we know about:
  - $\sin^2(x) + \cos^2(x) \equiv 1$
  - $1 + \cot^2(x) \equiv \operatorname{cosec}^2(x)$
  - $\tan^2(x) + 1 \equiv \sec^2(x)$
  - all the compound angle formulae, like $\sin(A + B) \equiv \sin(A)\cos(B) + \sin(B)\cos(A)$
  - all the double angle formulae, like $\sin(2A) \equiv 2\sin(A)\cos(A)$
  - ...and more.
- there are connections between trigonometrical functions. In a variety of situations we find that the following pairs of functions are connected:
  - $\sin^2(x)$ and $\cos^2(x)$
  - $\cot^2(x)$ and $\operatorname{cosec}^2(x)$
  - $\tan^2(x)$ and $\sec^2(x)$

  Not only are they connected by the Pythagorean identities, but they are also connected through differentiation and integration, and other ways as well. So when you see a $\operatorname{cosec}^2(x)$ in an equation, you should immediately think of $\cot^2(x)$, it's partner.

  This sort of thing can often give you clues as to what to do in a question.

2.2 Part II : Which Visualisation Method Shall We Use?

As I've suggested, in order not to lose solutions, we have to have some means of finding *all* the solutions to a trigonometrical equation, not just *the one* the calculator gives you.

At the time of writing I only know of two such methods: drawing a graph, and the *CAST* method. The *CAST* method is the one that is usually taught in school (why?), but my favourite is the drawing-the-graph method.

2.2.1 Drawing the Graph

If you are not familiar with the idea of using a graph to solve simultaneous equations, or how simultaneous equations can be used to solve a single equation, then check out Appendix **??**.

So now you know that it is possible to solve an equation by turning it into a pair of simultaneous equations, drawing a graph of the two functions, and finding out where they cross. So what's all this got to do with solving trigonometrical equations? Well, if you had this equation to solve

$$\sin(\theta) = \tfrac{1}{2}$$

then we can think of it as a pair of simultaneous equations

$$\begin{aligned} y &= \sin(\theta) \\ y &= \tfrac{1}{2} \end{aligned}$$

and use the graphical method to solve them:

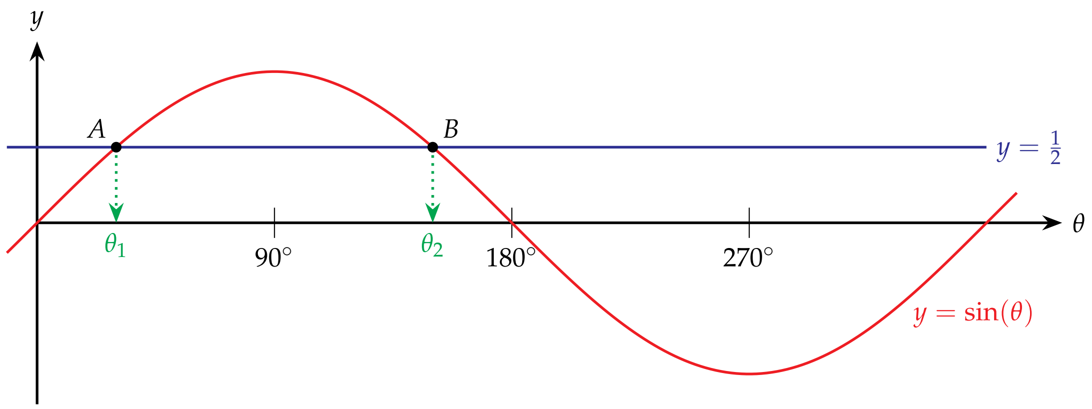


Figure 2: Solving $\sin(\theta) = \frac{1}{2}$

Now we've drawn the graphs, how do we find the solutions? Well, we use our calculator to find one solution:

$$\begin{aligned} \theta &= \sin^{-1}\left(\tfrac{1}{2}\right) \\ &= 30^\circ \end{aligned}$$

That must be θ_1, as that angle is between 0° and 90°. So how do we find θ_2? We use the symmetry of the graph! I'm hoping that you can see that

$$\begin{aligned} \theta_2 &= 180^\circ - \theta_1 \\ &= 180^\circ - 30^\circ \\ &= 150^\circ \end{aligned}$$

2.2.2 Drawing the Graph for More Trickier Problems

OK. Sofa so good.[1]

But what if you ended up (after completing Part I) with something like

$$\cos(2\theta - 45^\circ) = \tfrac{1}{2}$$

for $0^\circ \leq \theta \leq 360^\circ$? How do you draw a graph of $y = \cos(2\theta - 45^\circ)$? That's a cosine graph after a couple of nasty transformations! I don't fancy that!

Well, the good news is that you don't have to know how to draw the graph of $y = \cos(3\theta - 45^\circ)$. Here's what you do instead.

[1] I actually said this once to my wife after I'd repaired our old settee with string. She hit me.

As you know how to draw the graph of $y = \cos(x)$, then what we could do is use a cunning change of variable ploy:

$$\text{let } x = 2\theta - 45^\circ$$

because then we would have $\cos(x) = \frac{1}{2}$ to solve! And we know how to do that!

There is, however, one small fly in the ointment: since

$$0^\circ \leq \theta \leq 360^\circ$$

then

$$\begin{aligned} \implies 0^\circ &\leq 2\theta \leq 720^\circ \\ \implies -45^\circ &\leq 2\theta - 45^\circ \leq 675^\circ \\ \implies -45^\circ &\leq x \leq 675^\circ \end{aligned}$$

This means that we have to draw our graph over a much wider domain! But at least we only have to draw a graph of $y = \cos(x)$. Here goes:

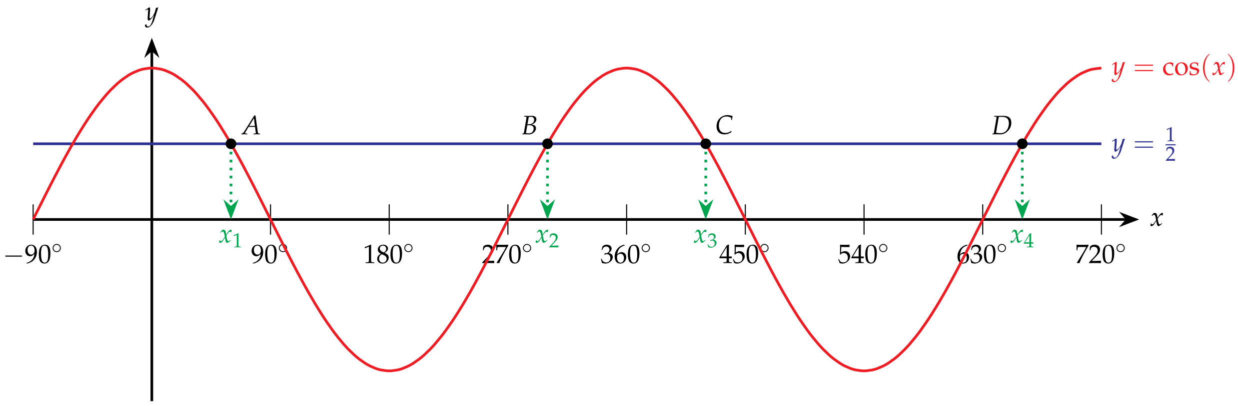


Figure 3: Solving $\cos(x) = \frac{1}{2}$

This time we have four solutions. How do we find them? As usual, we let the calculator give us one of them, and we use the symmetry of the cosine graph to find the rest:

$$\begin{aligned} x &= \cos^{-1}\left(\tfrac{1}{2}\right) \\ &= 60^\circ \end{aligned}$$

This must be x_1, as x_1 lies in the range $0^\circ - 90^\circ$. Now we can use the symmetry of the graph to find the rest:

$$\begin{aligned} x_2 &= 360^\circ - x_1 = 300^\circ \\ x_3 &= 360^\circ + x_1 = 420^\circ \\ x_4 &= 720^\circ - x_1 = 660^\circ \end{aligned}$$

Nearly done! Don't forget though that we need to find the value of θ! Since

$$x = 2\theta - 45^\circ$$

then

$$\theta = \tfrac{1}{2}\left(x + 45^\circ\right)$$

and so

$$\begin{aligned} \theta_1 &= \tfrac{1}{2}\left(60^\circ + 45^\circ\right) = 52.5^\circ \\ \theta_2 &= \tfrac{1}{2}\left(300^\circ + 45^\circ\right) = 172.5^\circ \\ \theta_3 &= \tfrac{1}{2}\left(420^\circ + 45^\circ\right) = 232.5^\circ \\ \theta_4 &= \tfrac{1}{2}\left(660^\circ + 45^\circ\right) = 352.5^\circ \end{aligned}$$

2.2.3 The *CAST* Diagram

In order not to interrupt the flow too much, I've shoved a discussion of the *CAST* diagram into Appendix B. One of the reasons for that is that I don't use it, and I would encourage you not to use it either!

3 Examples

3.1 Example 1: Solve $\tan(\theta) = 2$

Solve

$$\tan(\theta) = 2$$

for $0^\circ \leq \theta \leq 360^\circ$.

3.1.1 Part I

Well, there's nothing to do for Part I! We can go straight to Part II!

3.1.2 Part II

Drawing the Graph We now have to draw the graph of $y = \tan(\theta)$ and $y = 2$. We find where the two lines meet, as outlined in Section 1.3.2.

Here's my picture for this problem:

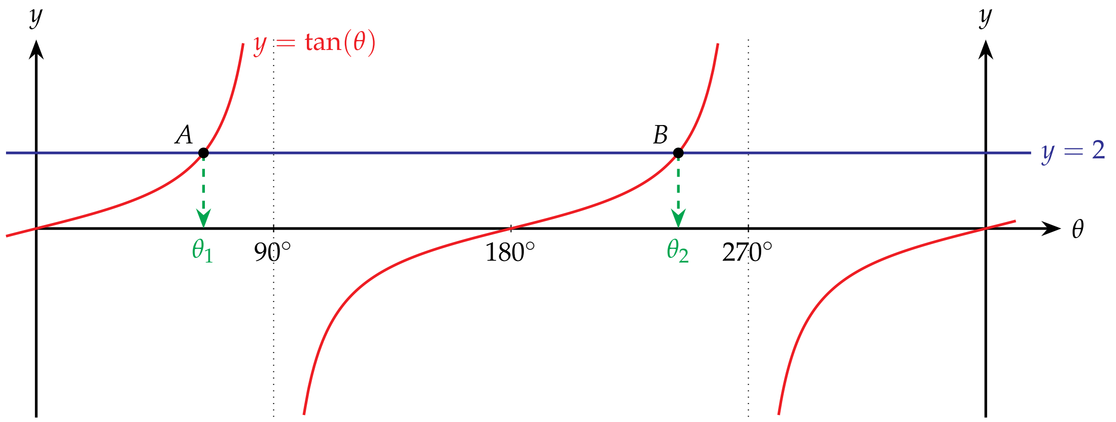


Figure 4: Solving $\tan(\theta) = 2$

Using a calculator for

$$\theta = \tan^{-1}(2) = 63.4^\circ$$

to three significant figures. Is this one of our solutions? It must be θ_1, as that is between 0° and 90°.

And to find θ_2, we use the symmetry of the graph:

$$\theta_2 = \theta_1 + 180^\circ = 243.4^\circ$$

to three significant figures.

Using the Unit Circle Here's my unit circle diagram for the $\tan(\theta) = 2$ part of this problem.

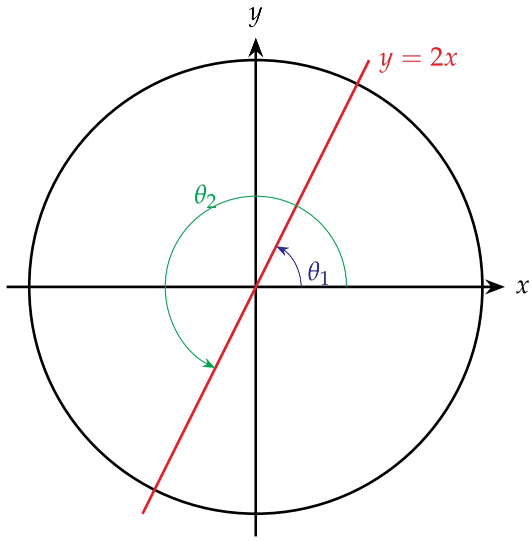


Figure 5: Solving $\tan(\theta) = 2$ Using the Unit Circle

The θ_1 and θ_2 angles correspond to those in the "Drawing the Graph" method above.

Notice that to solve problems with the tangent function, we have to find where a line with a given *gradient* crosses the unit circle. In this case, as the equation to solve is $\tan(\theta) = 2$, we have to draw the line with a gradient of 2.

3.2 Example 2: Edexcel C3, June 2005, Q1

(a) Given that $\sin^2(x) + \cos^2(x) \equiv 1$, show that $1 + \tan^2(x) \equiv \sec^2(x)$.
[2 marks]

(b) Solve, for $0^\circ \leq \theta < 360^\circ$, the equation

$$2\tan^2(\theta) + \sec(\theta) = 1 \tag{1}$$

giving your answers to 1 decimal place.
[6 marks]

3.2.1 Preliminaries...

It's part (b) of this question that's the "solve the trigonometrical equation" bit, but for this question we have other stuff to do first. This is common: very often you have preliminary parts to a question that will help you solve the trigonometrical equation when you get to it.

You very often get clues in a question that this is happening. The use of the word "hence", for example. Quite often in a multi-part question, as here, even if you don't have the word "hence" specifically mentioned, they are trying to guide you through the whole thing. So in multi-part questions, always cast your eyes on previous parts for clues on how to answer the current part.

Part (a) is actually something that you should have been exposed to in class. There is a standard technique to do this. Start with

$$\sin^2(x) + \cos^2(x) \equiv 1$$

then divide both sides of the equation by $\cos^2(x)$:

$$\frac{\sin^2(x) + \cos^2(x)}{\cos^2(x)} \equiv \frac{1}{\cos^2(x)}$$

split up the left-hand side into two fractions, using the way fractions add, in reverse:

$$\frac{\sin^2(x)}{\cos^2(x)} + \frac{\cos^2(x)}{\cos^2(x)} \equiv \frac{1}{\cos^2(x)}$$

and then use the definitions of $\tan(x)$ and $\sec(x)$:

$$\tan^2(x) + 1 \equiv \sec^2(x)$$

and we're done.

3.2.2 Part I

Right: now for part (b). At last we've got to the "solve the trigonometrical equation" bit. And what was I saying about using previous parts of the question to help you do something? Do you think that part (a) of this question is relevant here? Of course it is!

Using the result from part (a) we can convert the $\tan^2(\theta)$ bit of Equation 1 into $\sec^2(\theta)$ stuff! How can that help? Well, if we do that, we will get an equation with $\sec^2(\theta)$ in it, $\sec(\theta)$ in it, and a number in it. Remind you of anything?

So, starting with

$$2\tan^2(\theta) + \sec(\theta) = 1$$

if we use the result from part (a) we will get

$$2\left[\sec^2(\theta) - 1\right] + \sec(\theta) = 1$$

multiply out the brackets, and collect all the terms on the left

$$\begin{aligned} 2\sec^2(\theta) - 2 + \sec(\theta) &= 1 \\ \implies 2\sec^2(\theta) + \sec(\theta) - 3 &= 0 \end{aligned}$$

and blow me down with a feather! We've got a quadratic!

Factorising the quadratic we get

$$\left[\,2\sec(\theta)+3\,\right]\left[\,\sec(\theta)-1\,\right]=0$$

Now because we have two quantities that multiply together to give zero, then one of them must be zero. This means that

$$\text{either} \qquad 2\sec(\theta)+3=0 \qquad \text{or} \qquad \sec(\theta)-1=0$$

so

$$\text{either} \qquad \sec(\theta)=-\tfrac{3}{2} \qquad \text{or} \qquad \sec(\theta)=1$$

Remember what we have to get at the end of Part I of our cunning plan? So:

$$\text{either} \qquad \cos(\theta)=-\tfrac{2}{3} \qquad \text{or} \qquad \cos(\theta)=1$$

And that's the end of Part I. After the adverts, we can start watching Part II...

3.2.3 Part II

Drawing the Graph Here's my picture for this problem.

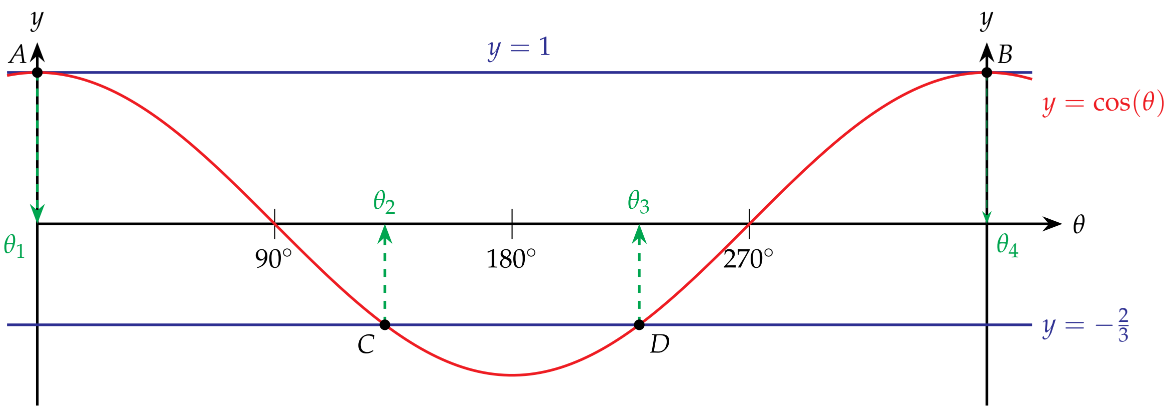


Figure 6: Edexcel C3, June 2005, Q1

θ_1 is clearly at 0°. We don't need the calculator for that!

θ_4 is clearly at 360°. We don't want this solution as it is not in our range of $0^\circ \le \theta < 360^\circ$.

Using a calculator for

$$\theta = \cos^{-1}\left(-\tfrac{2}{3}\right) = 131.8^\circ$$

to one decimal place. Is this one of our solutions? It must be θ_2, as that is between 90° and 180°.

And to find θ_3, we use the symmetry of the graph:

$$\theta_3 = 360^\circ - 131.8^\circ = 228.2^\circ$$

to one decimal place.

Using the Unit Circle Here's my unit circle diagram for this problem.

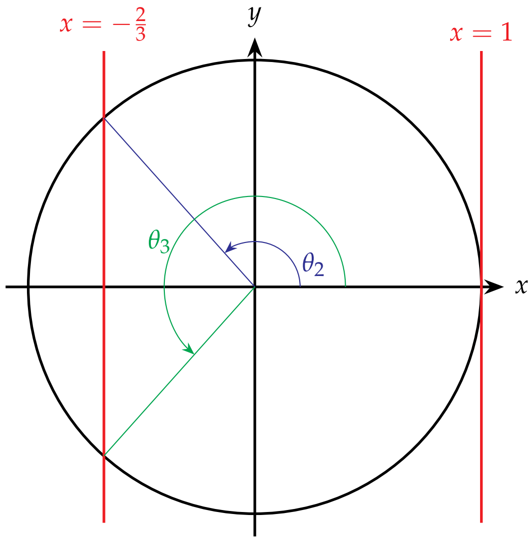


Figure 7: Solving $\cos(\theta) = -\frac{2}{3}$ and $\cos(\theta) = 1$ Using the Unit Circle

The θ_2 and θ_3 angles are the same as in the "Drawing the Graph" method above.

The θ_1 and θ_4 angles correspond to the solutions of $0°$ and $360°$ which are obtained from the intersection of $x = 1$ with the unit circle.

3.2.4 Lessons to be Learned

- Use what you've done in previous parts of a multi-part question.
- Watch out for hidden quadratics.
- Use of the $\sin^2(x) + \cos^2(x) \equiv 1$ identity.
- Use of the $\tan^2(x) + 1 \equiv \sec^2(x)$ identity.

3.3 Example 3: Edexcel C3, January 2006, Q6

$$f(x) = 12\cos(x) - 4\sin(x)$$

Given that $f(x) = R\cos(x + \alpha)$, where $R \geq 0$ and $0° \leq \alpha \leq 90°$,

(a) find the value of R and the value of α.
[4 marks]

(b) Hence solve, for $0° \leq x < 360°$, the equation

$$12\cos(x) - 4\sin(x) = 7 \tag{2}$$

giving your answers to 1 decimal place.
[5 marks]

(c) ...
[3 marks]

3.3.1 Preliminaries...

It's part (b) of this question that's the "solve the trigonometrical equation" bit, but for this question we have other stuff to do first. This is common: very often you have preliminary parts to a question that will help you solve the trigonometrical equation when you get to it.

You very often get clues in a question that this is happening. The use of the word "hence", for example. Quite often in a multi-part question, as here, even if you don't have the word "hence" specifically mentioned, they are trying to guide you through the whole thing. So in multi-part questions, always cast your eyes on previous parts for clues on how to answer the current part.

Part (a) is a type of question that very often comes up on exam papers. And there is a recipe for it. As you do the same thing every time with this kind of question, you should be picking up all the marks for these.

Here we go...

The plan is to compare $f(x)$ with the expanded form of the compound angle formula for the thing we have to get. Here's what I mean. Using the compound angle formula for $\cos(x + \alpha)$,

$$R\cos(x + \alpha) \equiv R\cos(\alpha)\cos(x) - R\sin(\alpha)\sin(x)$$

Now we compare that with $f(x)$:

$R\cos(\alpha)$	$\cos(x)$	$-$	$R\sin(\alpha)$	$\sin(x)$
12	$\cos(x)$	$-$	4	$\sin(x)$

Now because x is a variable, so it can take any value, then the only way that these two expressions are going to be the same *for all values of* x is if the respective coloured bits are the same. That is, when

$$R\sin(\alpha) = 4 \tag{3}$$

$$\text{and} \qquad R\cos(\alpha) = 12 \tag{4}$$

When you do this kind of question, you will *always* end up with this pair of simultaneous equations. The only difference from one question to another is what the numbers on the right-hand side will be. And that's still the case whether we had a question involving $R\cos(x + \alpha)$, $R\cos(x - \alpha)$, $R\sin(x + \alpha)$ or $R\sin(x - \alpha)$. We will *always* end up with the two simultaneous equations (3) and (4) above (with different numbers).

So you need to have a way of solving them!

There's a couple of ways of doing this. First, to find α we could divide (3) by (4) to give

$$\frac{R\sin(\alpha)}{R\cos(\alpha)} = \frac{4}{12}$$

Now on the left-hand side the Rs will cancel and we can use the definition of $\tan(x)$; on the right-hand side we can cancel top and bottom a bit:

$$\tan(\alpha) = \tfrac{1}{3}$$

so that

$$\alpha = \tan^{-1}\left(\tfrac{1}{3}\right) \approx 18.43^\circ$$

To find R *exactly* we square (3) and (4) and add:

$$R^2\sin^2(\alpha) + R^2\cos^2(\alpha) = 4^2 + 12^2$$

Factorising the left-hand side we get

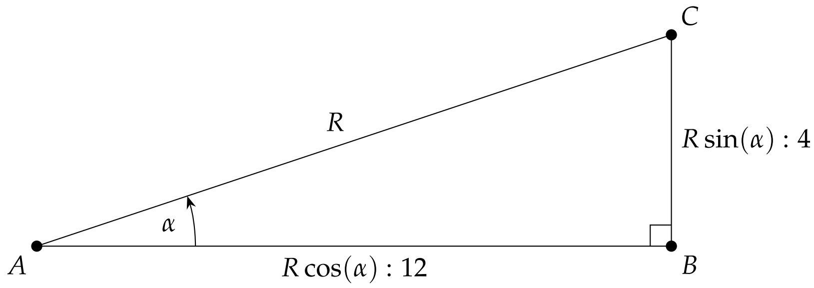
$$R^2\left[\sin^2(\alpha) + \cos^2(\alpha)\right] = 4^2 + 12^2$$

but $\sin^2(\alpha) + \cos^2(\alpha) \equiv 1$, so

$$R^2 = 4^2 + 12^2$$

and so $R = \sqrt{4^2 + 12^2} = 4\sqrt{10}$.

The other way of solving these equations is to draw the following right-angled triangle:



Start by drawing the triangle with angle α and hypotenuse R. From the definition of $\sin(\theta)$ and $\cos(\theta)$ (see Appendix B.1) then the side AB will be $R\cos(\alpha)$, and the side BC will be $R\sin(\alpha)$.

But we know that $R\cos(\alpha) = 12$ and $R\sin(\alpha) = 4$. So we can use Pythagoras to find R:

$$R = \sqrt{4^2 + 12^2} = 4\sqrt{10}$$

as before, and we can use the definition of $\tan(\alpha)$ to get

$$\tan(\alpha) = \tfrac{4}{12}$$

so that

$$\alpha = \tan^{-1}\left(\tfrac{1}{3}\right) \approx 18.43^\circ$$

as before.

3.3.2 Part I

Right: now for part (b). At last we've got to the "solve the trigonometrical equation" bit. And what was I saying about using previous parts of the question to help you do something? Do you think that part (a) of this question is relevant here? Of course it is!

Using the result from part (a) we can convert the $f(x)$ bit of Equation 2:

$$4\sqrt{10}\cos(x + \alpha) = 7$$

where $\alpha \approx 18.43^\circ$. Divide both sides by $4\sqrt{10}$:

$$\cos(x + \alpha) = \frac{7}{4\sqrt{10}}$$

and we're done for Part I!

3.3.3 Part II

Drawing the Graph Now trying to draw a picture of $\cos(x+\alpha)$ would involve a shift-transformation of the cosine graph. I don't like that! So, I'm going to use that cunning change of variable idea:

$$\text{let } \eta = x + \alpha$$

(η is my favouite greek letter!!) so the equation we have to solve is

$$\cos(\eta) = \frac{7}{4\sqrt{10}}$$

Here's my picture for this problem.

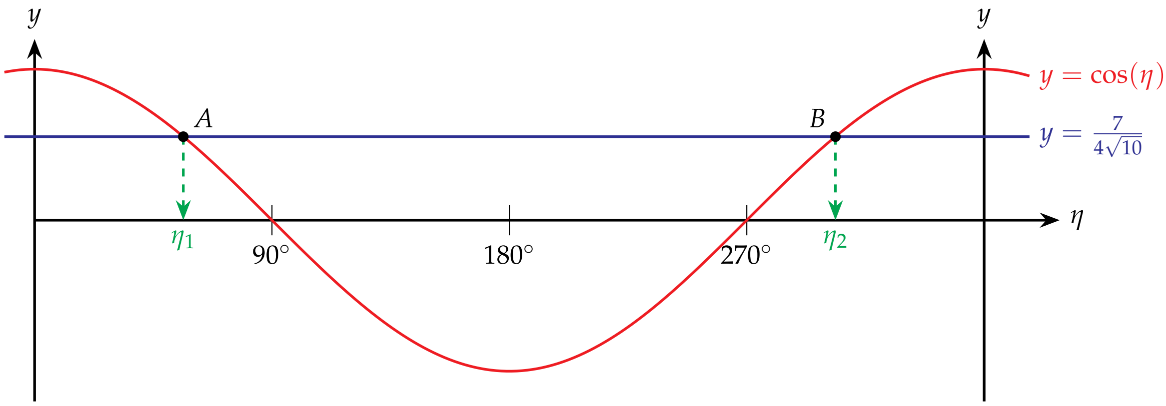


Figure 8: Edexcel C3, January 2006, Q6

Using a calculator for

$$\eta = \cos^{-1}\left(\frac{7}{4\sqrt{10}}\right) \approx 56.40^\circ$$

to two decimal places. Is this one of our solutions? It must be η_1, as that is between 0° and 90°.

And to find θ_2, we use the symmetry of the graph:

$$\eta_2 = 360^\circ - 56.40^\circ = 303.60^\circ$$

to two decimal places.

Now we know the two values of η, we can find the two values of x. Since $\eta = x + \alpha$, then

$$x = \eta - \alpha$$

so

$$x_1 = \eta_1 - \alpha = 56.40^\circ - 18.43^\circ = 38.0^\circ \text{ (to 1 dp)}$$
$$x_2 = \eta_2 - \alpha = 303.60^\circ - 18.43^\circ = 285.2^\circ \text{ (to 1 dp)}$$

Using the Unit Circle Here's my unit circle diagram for this problem.

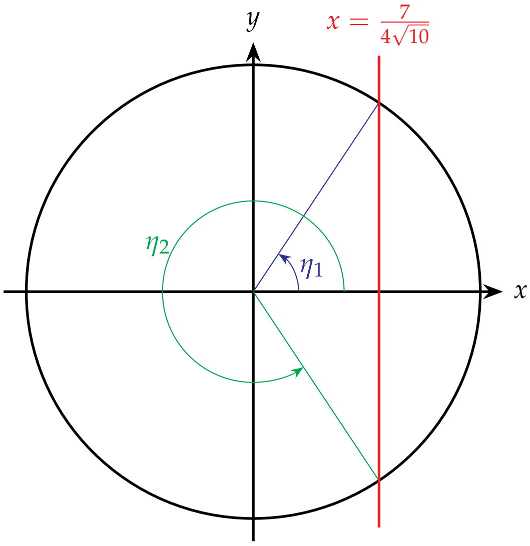


Figure 9: Solving $\cos(\eta) = \frac{7}{4\sqrt{10}}$ Using the Unit Circle

The η_1 and η_2 angles correspond to those in the "Drawing the Graph" method above.

3.3.4 Lessons to be Learned

- Use what you've done in previous parts of a multi-part question.
- Use of the compound-angle formulae.
- Using the cunning change-of-variable ploy.
- You *must* have a technique for solving the simultaneous equations (3) and (4). This kind of question *will* come up on your exam, so you must have a way of solving these equations quickly, without having to think too much. This is something you can prepare for.

3.4 Example 4: OCR C3, January 2006, Q9

(i) By first writing $\sin(3\theta)$ as $\sin(2\theta + \theta)$, show that

$$\sin(3\theta) \equiv 3\sin(\theta) - 4\sin^3(\theta) \tag{5}$$

[4 marks]

(ii) ...
[3 marks]

(iii) Solve, for $0° < \beta < 90°$, the equation

$$3\sin(6\beta)\text{cosec}(2\beta) = 4 \tag{6}$$

[6 marks]

3.4.1 Preliminaries...

It's part (iii) of this question that's the "solve the trigonometrical equation" bit, but for this question we have other stuff to do first. This is common: very often you have preliminary parts to a question that will help you solve the trigonometrical equation when you get to it.

You very often get clues in a question that this is happening. The use of the word "hence", for example. Quite often in a multi-part question, as here, even if you don't have the word "hence" specifically mentioned, they are trying to guide you through the whole thing. So in multi-part questions, always cast your eyes on previous parts for clues on how to answer the current part.

(i) As suggested, let's start by writing $\sin(3\theta)$ as $\sin(2\theta + \theta)$:

$$\begin{aligned}\sin(3\theta) &\equiv \sin(2\theta + \theta) \\ &\equiv \sin(2\theta)\cos(\theta) + \cos(2\theta)\sin(\theta)\end{aligned}$$

using a compound-angle formula. Now what? Well, comparing this with where we need to get to, we see that on the right-hand side of (5) there are only single angles (θ), whereas we have 2θ angles in both the $\sin(2\theta)$ and $\cos(2\theta)$ functions.

Well this gives me an idea! We could use the compound-angle formulae again to turn these double-angles into single angles. The problem is, of course, that $\cos(2\theta)$ has *three* ways of expanding it! Bum. How will we know which one to use?

Let's not worry about that just yet. Let's work on the $\sin(2\theta)$ bit first. Since $\sin(2x) \equiv 2\sin(x)\cos(x)$, then

$$\begin{aligned}\sin(3\theta) &\equiv 2\sin(\theta)\cos(\theta)\cos(\theta) + \cos(2\theta)\sin(\theta) \\ &\equiv 2\sin(\theta)\cos^2(\theta) + \cos(2\theta)\sin(\theta)\end{aligned}$$

Ooh! Hang on! Looking at the right-hand side of (5) again, I notice that we only have sin(...) stuff. No cos(...) stuff. So how can we get rid of our $\cos^2(\theta)$? Oh of course! We can use $\sin^2(\theta) + \cos^2(\theta) \equiv 1$:

$$\begin{aligned}\sin(3\theta) &\equiv 2\sin(\theta)\left[1 - \sin^2(\theta)\right] + \cos(2\theta)\sin(\theta) \\ &\equiv 2\sin(\theta) - 2\sin^3(\theta) + \cos(2\theta)\sin(\theta)\end{aligned}$$

Hey that's good! We have a $\sin^3(\theta)$ thing crop up! We must be on the right lines!

Just the $\cos(2\theta)$ thing to worry about now. Looking at where we need to get to again I suddenly realise that we only want sin(...) stuff. And I've just realised that there is a version of the $\cos(2x)$ thing that's only got sin(...) stuff in it. Let's try that: since $\cos(2x) \equiv 1 - \sin^2(x)$, then

$$\begin{aligned}\sin(3\theta) &\equiv 2\sin(\theta) - 2\sin^3(\theta) + \left[1 - 2\sin^2(\theta)\right]\sin(\theta) \\ &\equiv 2\sin(\theta) - 2\sin^3(\theta) + \sin(\theta) - 2\sin^3(\theta) \\ &\equiv 3\sin(\theta) - 4\sin^3(\theta)\end{aligned}$$

How about that!

3.4.2 Part I

Right: now for part (iii). At last we've got to the "solve the trigonometrical equation" bit. And what was I saying about using previous parts of the question to help you do something? Do you think that part (i) of this question is relevant here? Of course it is!

Using the result from part (i) we can see that

$$\sin(6\beta) \equiv 3\sin(2\beta) - 4\sin^3(2\beta)$$

Can you see that? All I've done is to replace the θ in Equation (5) by 2β. That means that Equation (6) can be written as:

$$3\big[3\sin(2\beta) - 4\sin^3(2\beta)\big]\text{cosec}(2\beta) = 4$$

Now $\text{cosec}(x) \equiv \dfrac{1}{\sin(x)}$, so:

$$\frac{3\big[3\sin(2\beta) - 4\sin^3(2\beta)\big]}{\sin(2\beta)} = 4$$

Oooh! I feel a bit of cancelling coming on! First, multiply out the brackets on the top:

$$\frac{9\sin(2\beta) - 12\sin^3(2\beta)}{\sin(2\beta)} = 4$$

Factorise a $\sin(2\beta)$ out of the top:

$$\frac{\sin(2\beta)\big[9 - 12\sin^2(2\beta)\big]}{\sin(2\beta)} = 4$$

And now we can cancel the $\sin(2\beta)$ from the top and the bottom of the left-hand side:

$$9 - 12\sin^2(2\beta) = 4$$

Now we need to make the $\sin^2(\beta)$ the subject of this:

$$\begin{aligned} \implies 9 &= 4 + 12\sin^2(2\beta) \\ \implies 5 &= 12\sin^2(2\beta) \\ \implies \tfrac{5}{12} &= \sin^2(2\beta) \end{aligned}$$

And finally, square-rooting both sides gives:

$$\implies \sin(2\beta) = \pm\sqrt{\tfrac{5}{12}}$$

remembering the possibility of the plus-or-minus thing.

And that's the end of Part I !!!

3.4.3 Part II

Drawing the Graph Now trying to draw a picture of $\sin(2\beta)$ would be far, far, too difficult. I couldn't possibly attempt that! So, I'm going to use that cunning change of variable idea:

$$\text{let } \eta = 2\beta$$

so the equation we have to solve is

$$\sin(\eta) = \pm\sqrt{\tfrac{5}{12}}$$

Don't forget the small fly in the ointment: since in this question,

$$0^\circ \leq \beta \leq 90^\circ$$

then

$$\begin{aligned} \implies 0^\circ &\leq 2\beta \leq 180^\circ \\ \implies 0^\circ &\leq \eta \leq 180^\circ \end{aligned}$$

so we're only interested in finding values of η between 0° and 180°. Here's my picture for this problem:

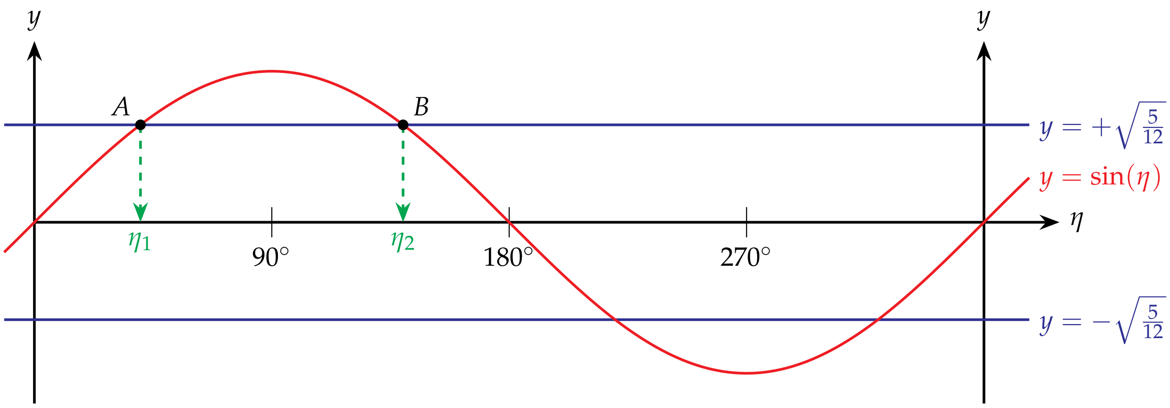


Figure 10: OCR C3, January 2006, Q9

Using a calculator for

$$\eta = \sin^{-1}\left[\sqrt{\tfrac{5}{12}}\right] \approx 40.20^\circ$$

to two decimal places. Is this one of our solutions? It must be η_1, as that is between 0° and 90°.

And to find η_2, we use the symmetry of the graph:

$$\eta_2 = 180^\circ - 40.20^\circ = 139.80^\circ$$

to two decimal places.

Now we know the two values of η, we can find the two values of β. Since $\eta = 2\beta$, then

$$\beta = \tfrac{1}{2}\eta$$

so

$$\begin{aligned} \beta_1 &= \tfrac{1}{2}\eta_1 = 20.10^\circ \text{ (to 2 dp)} \\ \beta_2 &= \tfrac{1}{2}\eta_2 = 69.90^\circ \text{ (to 2 dp)} \end{aligned}$$

Using the Unit Circle Here's my unit circle diagram for this problem.

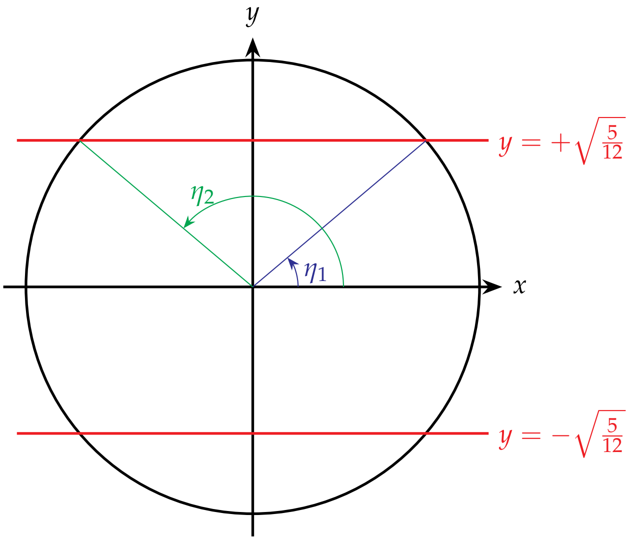


Figure 11: Solving $\sin(\eta) = \pm\sqrt{\frac{5}{12}}$ Using the Unit Circle

The η_1 and η_2 angles correspond to those in the "Drawing the Graph" method above. As η has to be between $0°$ and $180°$, then the places where $y = -\sqrt{\frac{5}{12}}$ intersect the unit circle cannot yield solutions that we want.

3.4.4 Lessons to be Learned

- Use what you've done in previous parts of a multi-part question.
- Use of the compound-angle formulae.
- Using the cunning change-of-variable ploy.
- Use of the $\sin^2(x) + \cos^2(x) \equiv 1$ identity.
- Always keep a look out for where you are trying to get to when you have a "prove the identity" type question.

3.5 Example 5: Solve $\sin(\theta) = \cos\left(\frac{1}{2}\theta\right)$

Solve

$$\sin(\theta) = \cos\left(\tfrac{1}{2}\theta\right)$$

for $0° \leq \theta \leq 360°$.

3.5.1 Part I

Well, what do we do with this? There's something to do with a double angle here, as θ is double $\frac{1}{2}\theta$! But how can we use that?

One thing we might be able to do is to use the identity

$$\sin(A+B) \equiv \sin(A)\cos(B) + \cos(A)\sin(B)$$

But how? We could be cunning! We could set $A = B = \frac{1}{2}\theta$, so that

$$\begin{aligned}\sin(\theta) = \sin\left(\tfrac{1}{2}\theta + \tfrac{1}{2}\theta\right) &= \sin\left(\tfrac{1}{2}\theta\right)\cos\left(\tfrac{1}{2}\theta\right) + \cos\left(\tfrac{1}{2}\theta\right)\sin\left(\tfrac{1}{2}\theta\right)\\ &= 2\sin\left(\tfrac{1}{2}\theta\right)\cos\left(\tfrac{1}{2}\theta\right)\end{aligned}$$

Aha! So our equation would become

$$2\sin\left(\tfrac{1}{2}\theta\right)\cos\left(\tfrac{1}{2}\theta\right) = \cos\left(\tfrac{1}{2}\theta\right)$$

It's tempting here to divide both sides by $\cos\left(\frac{1}{2}\theta\right)$, but that would be a mistake. Do you know why?? No? I'll tell you later! Instead, we should subtract $\cos\left(\frac{1}{2}\theta\right)$ from both sides

$$2\sin\left(\tfrac{1}{2}\theta\right)\cos\left(\tfrac{1}{2}\theta\right) - \cos\left(\tfrac{1}{2}\theta\right) = 0$$

and factorise

$$\cos\left(\tfrac{1}{2}\theta\right)\left[2\sin\left(\tfrac{1}{2}\theta\right) - 1\right] = 0$$

Now because we have here two things that multiply together to give zero, then one of them must be zero. So either

$$\cos\left(\tfrac{1}{2}\theta\right) = 0 \qquad (7)$$

or

$$2\sin\left(\tfrac{1}{2}\theta\right) - 1 = 0$$

in which case

$$\sin\left(\tfrac{1}{2}\theta\right) = \tfrac{1}{2} \qquad (8)$$

and Equations (7) and (8) are the culmination of Part I.

3.5.2 Part II

Drawing the Graph We now have to draw *two* graphs, one containing $y = \cos\left(\frac{1}{2}\theta\right)$ and $y = 0$, and the other containing $y = \sin\left(\frac{1}{2}\theta\right)$ and $y = \frac{1}{2}$. We find where the two lines on each graph meet, as outlined in Section 1.3.2.

Let's start with $\cos\left(\frac{1}{2}\theta\right) = 0$. Now trying to draw a picture of $\cos\left(\frac{1}{2}\theta\right)$ would be far, far, too difficult. I couldn't possibly attempt that! So, I'm going to use that cunning change of variable idea:

$$\text{let } \eta = \tfrac{1}{2}\theta$$

so the equation we have to solve is

$$\cos(\eta) = 0$$

Don't forget the small fly in the ointment: since in this question,

$$0^\circ \leq \theta \leq 360^\circ$$

then

$$\implies 0^\circ \leq \tfrac{1}{2}\theta \leq 180^\circ$$
$$\implies 0^\circ \leq \eta \leq 180^\circ$$

so we're only interested in finding values of η between 0° and 180°. Here's my picture for this problem:

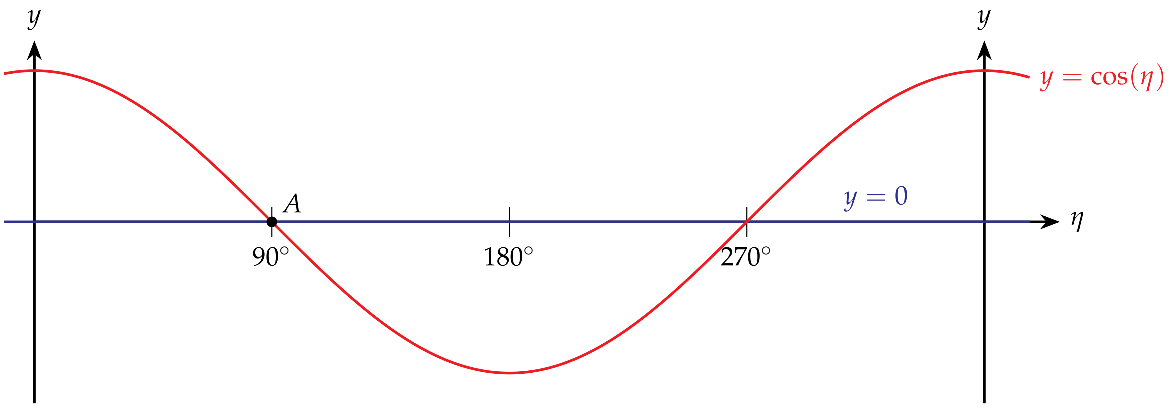


Figure 12: Solving $\cos(\eta) = 0$

So there's only one solution in the region in which we are interested: $\eta = 90^\circ$. That would mean that $\theta = 2\eta = 180^\circ$.

Right. The other equation we need to solve is $\sin\left(\frac{1}{2}\theta\right) = \frac{1}{2}$. Now trying to draw a picture of $\sin\left(\frac{1}{2}\theta\right)$ would be far, far, too difficult. I couldn't possibly attempt that! So, I'm going to use that cunning change of variable idea:

$$\text{let } \eta = \tfrac{1}{2}\theta$$

so the equation we have to solve is

$$\sin(\eta) = \tfrac{1}{2}$$

As before we're only interested in finding values of η between 0° and 180°. Here's my picture for this problem:

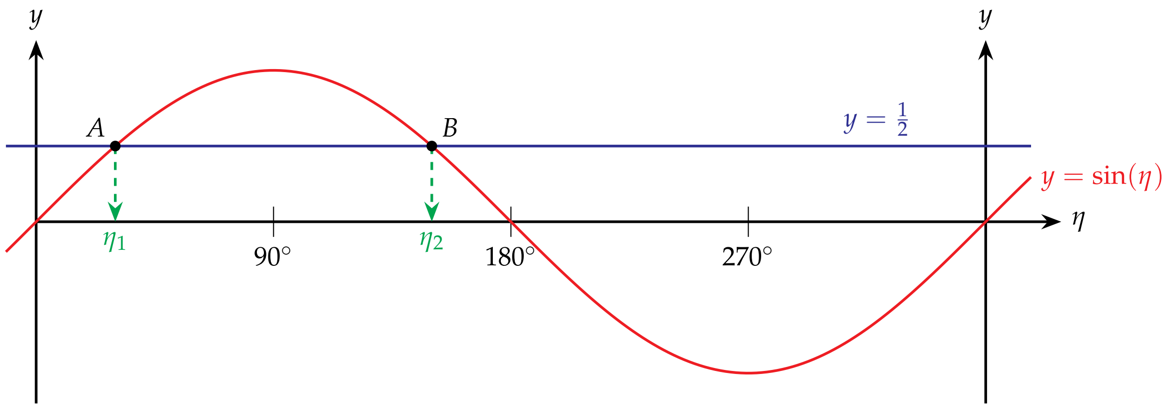


Figure 13: Solving $\sin(\eta) = \frac{1}{2}$

So there are two solutions in the region in which we are interested. Using a calculator for

$$\eta = \sin^{-1}\left(\tfrac{1}{2}\right) = 30^\circ$$

exactly. Is this one of our solutions? It must be η_1, as that is between 0° and 90°.

And to find η_2, we use the symmetry of the graph:

$$\eta_2 = 180^\circ - 30^\circ = 150^\circ$$

exactly.

Using the Unit Circle Here's my unit circle diagram for the $\sin(\eta) = \frac{1}{2}$ part of this problem.

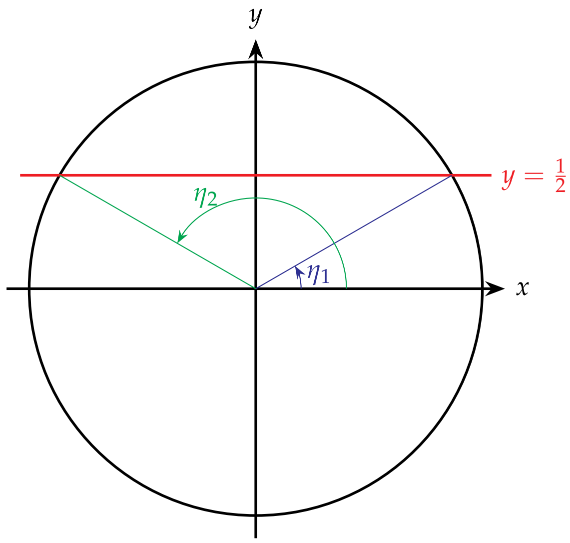


Figure 14: Solving $\sin(\eta) = \frac{1}{2}$ Using the Unit Circle

The η_1 and η_2 angles correspond to those in the "Drawing the Graph" method above.

3.5.3 The "Issue" from Part I

Do you remember that when we got to this equation in Part I

$$2\sin\left(\tfrac{1}{2}\theta\right)\cos\left(\tfrac{1}{2}\theta\right) = \cos\left(\tfrac{1}{2}\theta\right) \tag{9}$$

I said that it would be a mistake to divide both sides by $\cos\left(\frac{1}{2}\theta\right)$? Well, let's see why that would be. If we did divide both sides by $\cos\left(\frac{1}{2}\theta\right)$ we would get

$$2\sin\left(\tfrac{1}{2}\theta\right) = 1$$

which would lead to

$$\sin\left(\tfrac{1}{2}\theta\right) = \tfrac{1}{2}$$

Hang on a minute! Before, we ended up with *two* equations. Not only did we have the one above, but we also had $\cos\left(\frac{1}{2}\theta\right) = 0$ as well. What's happened to that??

What's happened is that when we divided both sides of Equation (9) by $\cos\left(\frac{1}{2}\theta\right)$, we were dividing both sides of an equation by a variable *which could be zero*! And that's not good!

So when we did the division, we should have taken into account that $\cos\left(\frac{1}{2}\theta\right)$ could be zero. Because of course, if $\cos\left(\frac{1}{2}\theta\right) = 0$, then Equation (9) would be satisfied, wouldn't it? Both sides of the equation would be 0. So if we forgot that $\cos\left(\frac{1}{2}\theta\right)$ could be zero, *we would lose solutions*.

Golden rule: be very careful when you divide both sides of an equation by a variable that could be zero!!

3.5.4 Lessons to be Learned

- Use of the compound-angle formulae.
- Using the cunning change-of-variable ploy.
- Don't divide both sides of an equation by zero.

3.6 Example 6: Edexcel C3, June 2009, Q2

(a) Use the identity $\cos^2(x) + \sin^2(x) \equiv 1$ to prove that $\tan^2(x) \equiv \sec^2(x) - 1$.
[2 marks]

(b) Solve, for $0° \leq \theta < 360°$, the equation

$$2\tan^2(\theta) + 4\sec(\theta) + \sec^2(\theta) = 2 \tag{10}$$

[6 marks]

3.6.1 Preliminaries...

It's part (b) of this question that's the "solve the trigonometrical equation" bit, but for this question we have other stuff to do first. This is common: very often you have preliminary parts to a question that will help you solve the trigonometrical equation when you get to it.

You very often get clues in a question that this is happening. The use of the word "hence", for example. Quite often in a multi-part question, as here, even if you don't have the word "hence" specifically mentioned, they are trying to guide you through the whole thing. So in multi-part questions, always cast your eyes on previous parts for clues on how to answer the current part.

Part (a) is actually something that you should have been exposed to in class. There is a standard technique to do this. Start with

$$\sin^2(x) + \cos^2(x) \equiv 1$$

then divide both sides of the equation by $\cos^2(x)$:

$$\frac{\sin^2(x) + \cos^2(x)}{\cos^2(x)} \equiv \frac{1}{\cos^2(x)}$$

split up the left-hand side into two fractions, using the way fractions add, in reverse:

$$\frac{\sin^2(x)}{\cos^2(x)} + \frac{\cos^2(x)}{\cos^2(x)} \equiv \frac{1}{\cos^2(x)}$$

and then use the definitions of $\tan(x)$ and $\sec(x)$:

$$\tan^2(x) + 1 \equiv \sec^2(x)$$

Now subtract 1 from both sides

$$\tan^2(x) \equiv \sec^2(x) - 1$$

and we're done.

3.6.2 Part I

Right: now for part (b). At last we've got to the "solve the trigonometrical equation" bit. And what was I saying about using previous parts of the question to help you do something? Do you think that part (a) of this question is relevant here? Of course it is!

Using the result from part (a) we can convert the $\tan^2(\theta)$ bit of Equation 10 into $\sec^2(\theta)$ stuff! How can that help? Well, if we do that, we will get an equation with $\sec^2(\theta)$ in it, $\sec(\theta)$ in it, and a number in it. Remind you of anything?

So, starting with

$$2\tan^2(\theta) + 4\sec(\theta) + \sec^2(\theta) = 2$$

if we use the result from part (a) we will get

$$2\big[\sec^2(\theta) - 1\big] + 4\sec(\theta) + \sec^2(\theta) = 2$$

multiply out the brackets, and collect all the terms on the left

$$2\sec^2(\theta) - 2 + 4\sec(\theta) + \sec^2(\theta) = 2$$
$$\implies 3\sec^2(\theta) + 4\sec(\theta) - 4 = 0$$

and blow me down with a feather! We've got a quadratic!

Factorising the quadratic we get

$$\big[\,3\sec(\theta) - 2\,\big]\,\big[\,\sec(\theta) + 2\,\big] = 0$$

Now because we have two quantities that multiply together to give zero, then one of them must be zero. This means that

$$\text{either} \qquad 3\sec(\theta) - 2 = 0 \qquad \text{or} \qquad \sec(\theta) + 2 = 0$$

so

$$\text{either} \qquad \sec(\theta) = \tfrac{2}{3} \qquad \text{or} \qquad \sec(\theta) = -2$$

Remember what we have to get at the end of Part I of our cunning plan? So:

$$\text{either} \qquad \cos(\theta) = \tfrac{3}{2} \qquad \text{or} \qquad \cos(\theta) = -\tfrac{1}{2}$$

And that's the end of Part I. After the adverts, we can start watching Part II...

3.6.3 Part II

Drawing the Graph Here's my picture for this problem.

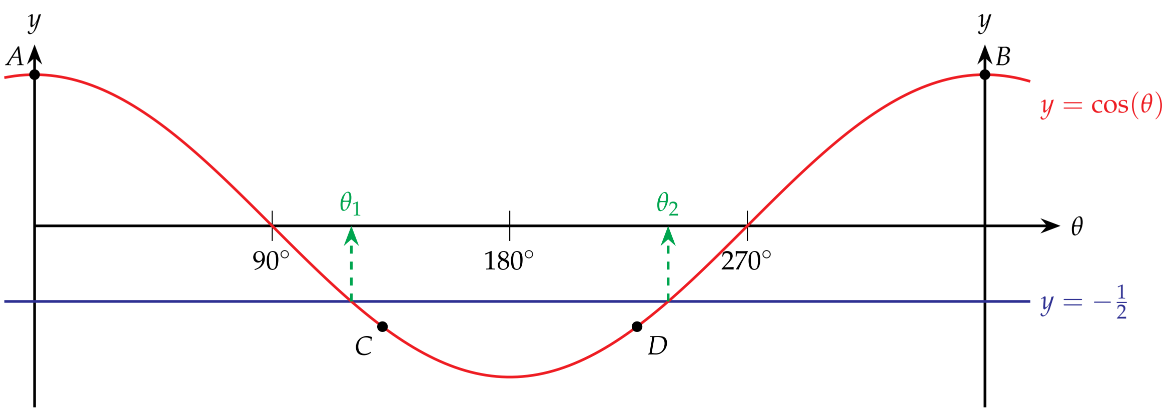


Figure 15: Edexcel C3, June 2009, Q2

Why is there only one horizontal blue line? What happened to the one that is supposed to represent $\cos(\theta) = \frac{3}{2}$? Oh yeah! $\cos(\theta)$ only goes up to 1, so it can't be $\frac{3}{2}$! So the equation $\cos(\theta) = \frac{3}{2}$ doesn't have any solutions.

Using a calculator for

$$\theta = \cos^{-1}\left(-\tfrac{1}{2}\right) = 120^\circ$$

exactly. Is this one of our solutions? It must be θ_1, as that is between 90° and 180°.

And to find θ_2, we use the symmetry of the graph:

$$\theta_2 = 360^\circ - 120^\circ = 240^\circ$$

to one decimal place.

Using the Unit Circle Here's my unit circle diagram for this problem.

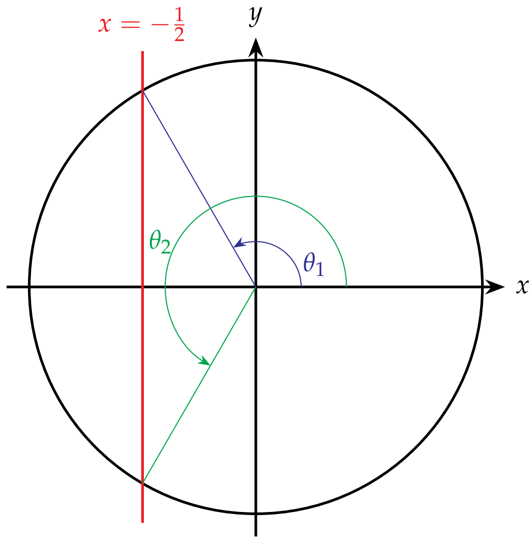


Figure 16: Solving $\cos(\theta) = -\frac{1}{2}$ Using the Unit Circle

The θ_1 and θ_2 angles are the same as in the "Drawing the Graph" method above.

3.6.4 Lessons to be Learned

- Use what you've done in previous parts of a multi-part question.
- Watch out for hidden quadratics.
- Use of the $\sin^2(x) + \cos^2(x) \equiv 1$ identity.
- Use of the $\tan^2(x) + 1 \equiv \sec^2(x)$ identity.

A Solving Simultaneous Equations...

A.1 ...Using Algebra...

When you have a pair of simultaneous equations, such as

$$y = x^2 \tag{11}$$
$$y = x + 2 \tag{12}$$

then one of the ways that we can solve them is by using algebra. We could, for example, use the *substitution* idea: from Equation (11) we know that $y = x^2$, so wherever we have a y we could write x^2, because they are the same thing.

Oh! We have a y in Equation (12)! So we can substitute x^2 for it:

$$x^2 = x + 2$$

and we have a quadratic, so we can use standard quadratic-solving techniques to solve it. That will give us two values for x: and we use those to find the two corresponding values for y.

A.2 ...Or Using a Graph

Another way of solving a pair of simultaneous equations is by a graphical method. I'm hoping that you've come across this idea already, but if not, here's how it works.

What you have to do is to draw the graphs of the two functions, in this case $y = x^2$ and $y = x + 2$, on the same set of axes. See Figure 17. Then you find out where they cross. And the points where the two functions cross are the solutions to the simultaneous equations!

Why is this? Well, if you think about it, the points marked A and B on Figure 17 are the places that lie on both lines. That means that at A and B, then $y = x^2$ (beacuse those points lie on the blue line) *and* $y = x + 2$ (as they lie on the red line as well). Well, these are the conditions for the simulataneous equations, aren't they?

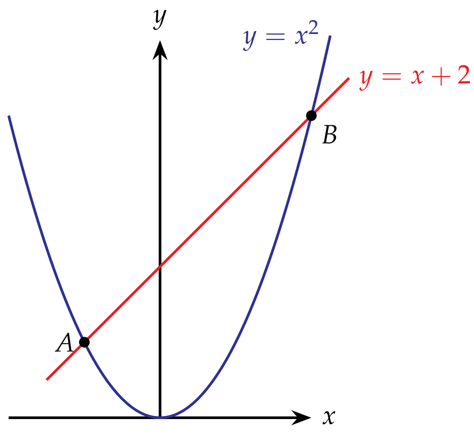


Figure 17: Solving Simultaneous Equations Graphically

So...solving a pair of simultaneous equations is the same thing as finding where two lines cross. It's the same problem, solved in two ways: one using algebra, the other using pictures.

This means that if we wanted to solve the equation

$$x^2 = x + 2$$

then one of the ways is to draw the lines $y = x^2$ and $y = x + 2$ on the same set of axes, and find out where they cross.

B The *CAST* Diagram

Before we get to the *CAST* diagram, I want to show you something else.

B.1 The Definition of $\sin(\theta)$ and $\cos(\theta)$

Have a look at Figure 18. This right-angled triangle with a hypotenuse of 1 can be used to actually *define* the $\sin(\theta)$

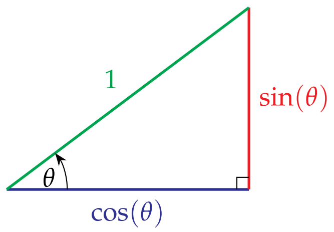


Figure 18: The Definition of $\sin(\theta)$ and $\cos(\theta)$

and $\cos(\theta)$ functions. Or, if you prefer, you could use the miserable *SOH CAH TOA*[2] monstrosity to obtain the lengths of the shorter two sides, given the angle θ and the length of the long side. Either way, Figure 18 is worth remembering.

B.2 The Unit Circle

Right. Now have a look at Figure 19. In this picture I've placed the triangle from Figure 18 into a circle that has a

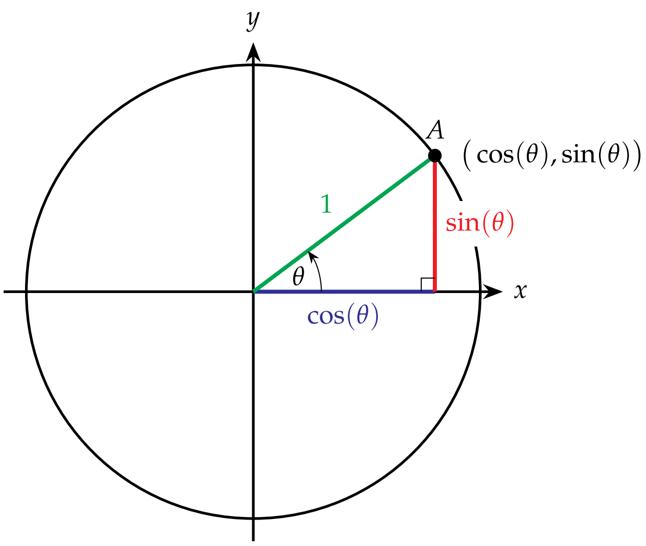


Figure 19: The Unit Circle

radius of 1. This is known as "the unit circle".

I'm hoping that you can see that the coordinates of point A in Figure 19 will be $\big(\cos(\theta), \sin(\theta)\big)$. That should be apparent as a consequence of Figure 18.

I've drawn this picture because from it we can see that for any point on the circle, $\cos(\theta)$ is equivalent to the point's x-coordinate, and $\sin(\theta)$ is equivalent to the point's y-coordinate.

So why is this useful?

Two reasons:

- it makes it easy to determine the signs of the sines and cosines of various angles;
- you can use this unit circle to show and find the solutions of trigonometrical equations.

[2] As you might guess, I'm not a fan of *SOH CAH TOA* as a teaching aid.

B.3 The Signs of $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$

If $\cos(\theta)$ is equivalent to the x-coordinate, then $\cos(\theta)$ will be positive when x is positive; and $\cos(\theta)$ will be negative when x is negative.

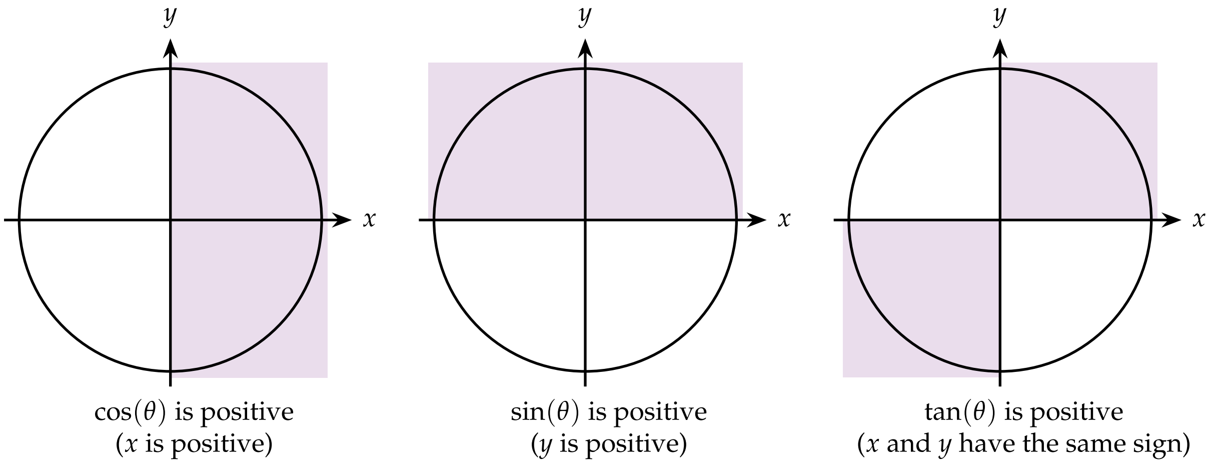


Figure 20: Where the Trigonometrical Functions are Positive

And if $\sin(\theta)$ is equivalent to the y-coordinate, then $\sin(\theta)$ will be positive when y is positive; and $\sin(\theta)$ will be negative when y is negative.

And because $\tan(\theta)$ is defined as $\tan(\theta) \equiv \dfrac{\sin(\theta)}{\cos(\theta)}$ then whenever $\sin(\theta)$ and $\cos(\theta)$ have the same sign, then $\tan(\theta)$ will be positive.

B.4 Solving Trigonometrical Equations Using the Unit Circle

We can also use the unit circle to solve trigonometrical equations, as an alternative to drawing the graphs of the functions.

B.4.1 Solving $\sin(\theta) = ...$ Problems

For example, if we ended up at the end of Part I with the equation $\sin(\theta) = \frac{1}{2}$ to solve, then we could draw the following unit circle and line:

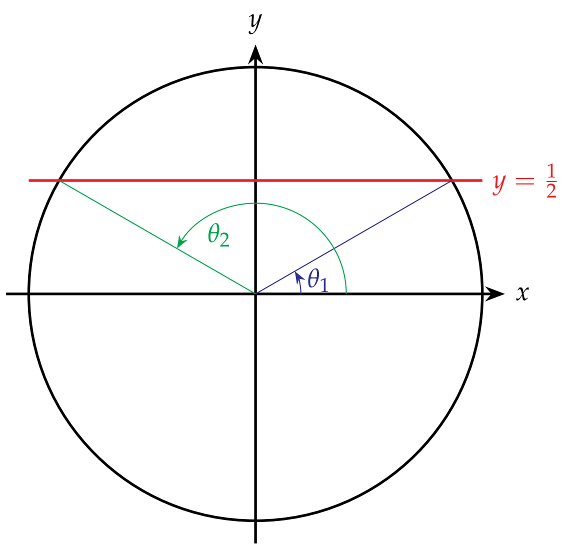


Figure 21: Solving $\sin(\theta) = \frac{1}{2}$ Using the Unit Circle

As $\sin(\theta)$ is equivalent to the y-coordinate, then the angles θ that we are interested in will be those which have a y-coordinate of $\frac{1}{2}$.

It's not as obvious here as it would be if we drew the graph of the $\sin(\theta)$ function, that the angles θ_1 and θ_2 shown in Figure 21 will not be the only solutions to $\sin(\theta) = \frac{1}{2}$. There will also be $\theta_1 + 360°$, $\theta_2 + 360°$, $\theta_1 + 720°$, $\theta_2 - 360°$, etc., etc.

It is because it's not so obvious using the unit circle how many solutions there will be for a trigonometrical equation, that I prefer to draw the graph of the $\sin(\theta)$ function over the range we are interested in. It's really easy then to see how many solutions there are, and how to use the symmetry of the graph to find them.

B.4.2 Solving $\cos(\theta) = \ldots$ Problems

If we ended up at the end of Part I with the equation $\cos(\theta) = \frac{1}{2}$ to solve, then we could draw the following unit circle and line:

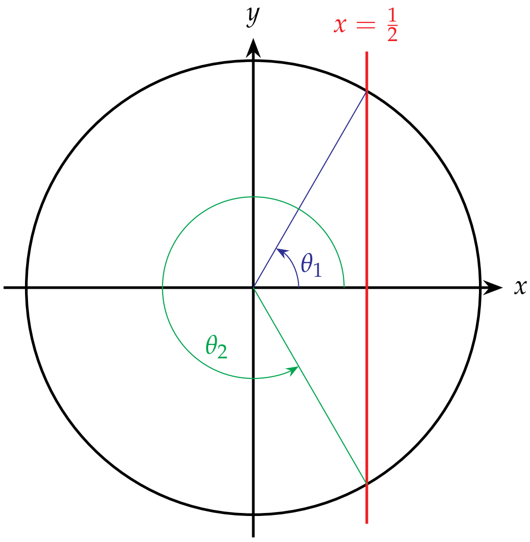


Figure 22: Solving $\cos(\theta) = \frac{1}{2}$ Using the Unit Circle

As $\cos(\theta)$ is equivalent to the x-coordinate, then the angles θ that we are interested in will be those which have an x-coordinate of $\frac{1}{2}$.

It's not as obvious here as it was when we drew the graph of the $\cos(\theta)$ function, that the angles θ_1 and θ_2 shown in Figure 22 will not be the only solutions to $\cos(\theta) = \frac{1}{2}$. There will also be $\theta_1 + 360°$, $\theta_2 + 360°$, $\theta_1 + 720°$, $\theta_2 - 360°$, etc., etc.

B.4.3 Solving $\tan(\theta) = \ldots$ Problems

If we ended up at the end of Part I with the equation $\tan(\theta) = \frac{1}{2}$ to solve, how would we solve that??

Remember that the definition of $\tan(\theta)$ is

$$\tan(\theta) \equiv \frac{\sin(\theta)}{\cos(\theta)} \equiv \frac{y}{x}$$

(because $\cos(\theta)$ is equivalent to the x-coordinate, and $\sin(\theta)$ is equivalent to the y-coordinate). That means that $\tan(\theta)$ is the *gradient* of a straight line drawn through the origin!

So then we could draw the unit circle and line shown in Figure 23.

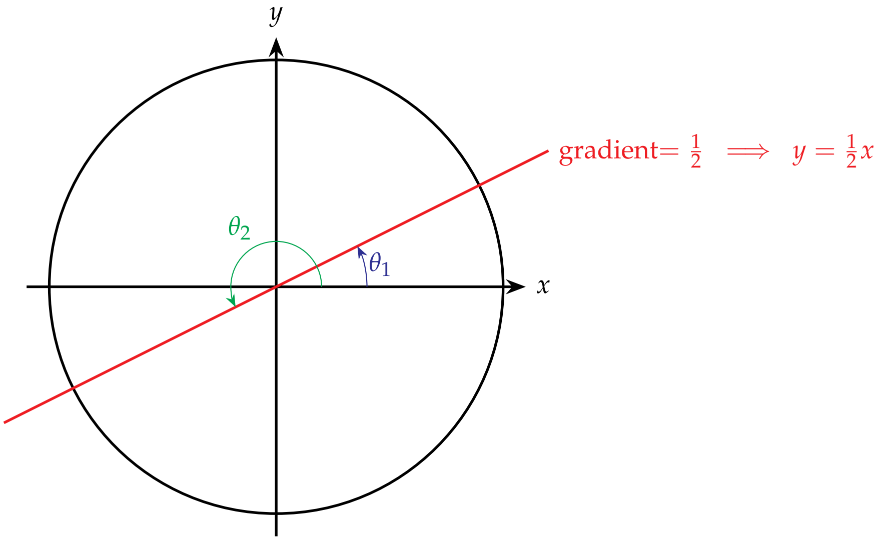


Figure 23: Solving $\tan(\theta) = \frac{1}{2}$ Using the Unit Circle

This time we are interested in the angles where the unit circle and a line through the origin with a gradient of $\frac{1}{2}$ meet.

Notice that because this time we are drawing a straight line through the origin, the points where the line crosses the circle will be 180° apart. And that's just what we find when we draw the graph of the $\tan(\theta)$ function to solve this equation!

But it's still not as obvious here as it would be if we drew the graph of the $\tan(\theta)$ function, that the angles θ_1 and θ_2 shown in Figure 23 will not be the only solutions to $\tan(\theta) = \frac{1}{2}$. There will also be $\theta_1 + 360°$, $\theta_2 + 360°$, $\theta_1 + 720°$, $\theta_2 - 360°$, etc., etc.

B.4.4 Trickier Problems

And if we had something like

$$\cos(2\theta - 45°) = \frac{1}{2}$$

to solve, what do we do then?

As described in Section 2.2.2, what we do here is to use a cunning change of variable:

$$\text{let } x = 2\theta - 45°$$

so we then have the equation

$$\cos(x) = \frac{1}{2}$$

to solve.

Once we've found the solutions for x, using the unit circle diagram from Section B.4.2, then we use

$$\theta = \frac{1}{2}(x + 45°)$$

to get the solutions for θ.

B.5 The *CAST* Diagram

So where does the *CAST* diagram come in? Well, the *CAST* diagram is really just the unit circle that we have seen before with a mnemonic for remembering the signs of the trigonometrical functions in the four different quadrants. See Figure 24.

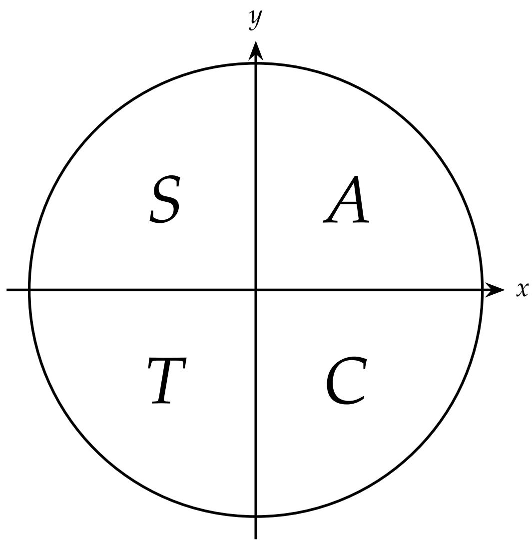


Figure 24: The *CAST* Diagram

The *C* represents the quadrant where only the cosine of angles is positive;

The *A* represents the quadrant where all three functions sine, cosine and tangent of angles are positive;

The *S* represents the quadrant where only the sine of angles is positive;

The *T* represents the quadrant where only the tangent of angles is positive.

To use this diagram, you have to have a mnemonic to remember the *CAST* cycle, such as:

- *A*ll *S*tations *T*o *C*helmsford
- *A*ntigen-*S*pecific *T*-*C*ells
- *C*oalition (to) *A*bolish *S*lavery *T*rafficking
- *S*teve *T*alks *C*rap *A*lways
- ...

Not so bad.

You have to have a way to remember which quadrant to start the naming of quadrants. This will be determined by your mnemonic. Not so bad.

You have to have a way of remembering which way to move around the circle as you name quadrants: clockwise or ant-clockwise. Not so bad.

But why bother *C*ramming *A*ll *T*his *S*hit??? If you remember Figure 18, that $\cos(\theta)$ is equivalent to the x-coordinate, and $\sin(\theta)$ is equivalent to the y-coordinate, then that's all you need. You can work out where things are positive or negative just by remembering where x and y are positive or negative.

Isn't that easier??