Steve Smith Tuition: Maths Notes

Percentages

Contents

Prerequisites

None.

Notes

None.

Document History

Date	Version	Comments
13th October 2012	1.0	Initial creation of the document.
26th November 2013	1.1	Including the non-calculator part and the "one quantity as a percentage of another" section.

Part I

If You Do Have a Calculator

1 Introduction

Questions on percentages can get confusing. When you're tackling a question on percentages, you often need to ask yourself things like:

- "Have we got a percentage increase or decrease?"
- "How do you express one quantity as a percentage of another?"
- "What is compound interest, and how do you calculate it?"
- "What are reversed percentages?"

Well, in this document I'm going to try and make questions on percentages as easy as possible. And I think that you can break most percentage questions down into one simple formula[1]:

Percentage Calculation Formula

The formula for doing percentage calculations is

$$new = old \times factor \quad (1)$$

Let me give you an example. Here's a question:

Question 1

A worker earns £6.80 an hour. One day he gets a 5% raise. How much does he get per hour now?

[1] The only exception is where we want to find one value as a percantage of another. See Section 7.

And here's the answer:

Answer 1

OK, to answer all percentages questions, we have to figure out what is going to be the *new* value, what is going to be the *old* value, and what the *factor* will be.

In this question, £6.80 is the *old* amount; the *new* amount is what we are trying to find, and the *factor* will be 1.05[a].

So, from

$$new = old \times factor$$

we can write

$$new = 6.80 \times 1.05 = 7.14$$

So the worker now earns (i.e. the *new* wage is) £7.14 an hour.

[a] I'm going to explain where this comes from very soon! Don't worry!

In this example, it's not too difficult to see that the *old* value is going to be the worker's old wage rate; and the *new* value will be the worker's new wage rate. But where does the *factor* come from?

For me, figuring out the value of the *factor* is the hardest thing you need to do in this method. And once you've got the hang of it, that's not hard at all.

1.1 Types of Percentage Questions

There are five kinds of percentage questions:

- percentage increase;
- percentage decrease;
- reversed percentages;
- compound interest, and repeated percentages;
- finding what one quantity is as a percentage of another.

We will be looking at each of the first four of these in turn, to see how we can use Equation(1) to solve them. The last needs a slightly different approach (see Section 7).

By way of a digression, after looking at percentage increase and decrease, I'm going to show you how the factor thing works. I'm hoping that by the time you get to Section 4, you will have an idea of how you come up with the factor yourself, but if you haven't, I'll show you how it works.

2 Percentage Increase

Here's the first percentage increase example:

Question 2

Increase 12 kg by 8%.

And here's the answer:

Answer 2

As before, we have to figure out what is going to be the *new* value, what is going to be the *old* value, and what the *factor* will be.

In this question, 12 kg is the *old* amount; the *new* amount is what we are trying to find, and the *factor* will be 1.08 this time (can you have a guess why?).

So, from

$$new = old \times factor$$

we can write

$$new = 12 \times 1.08 = 12.96\ kg$$

And here's another example:

Question 3

An advertisement for a breakfast cereal states that a special offer packet contains 15% more cereal for the same price than a normal 500 g packet. How much breakfast cereal is in the special offer packet?

And here's the answer to this one:

Answer 3

This time, 500 g is the *old* amount; the *new* amount is what we are trying to find, and the *factor* will be 1.15.

So, from

$$new = old \times factor$$

we can write

$$new = 500 \times 1.15 = 575\ g$$

So there 575 g of cereal in a special offer packet.

3 Percentage Decrease

Here's the first percentage decrease example:

Question 4

Decrease 350 m by 3%.

And here's the answer:

Answer 4

In this question, 350 m is the *old* amount; the *new* amount is what we are trying to find, and the *factor* will be 0.97 this time (can you have a guess why?).

So, from

$$new = old \times factor$$

we can write

$$new = 350 \times 0.97 = 339.5\ m$$

And here's another percentage decrease example:

Question 5

You are a member of a club which allows you to claim a 12% discount off any marked price in shops. What will you pay in total for the following goods?

- A sweatshirt, marked at £19;
- A tracksuit, marked at £26.

And here's the answer:

Answer 5

Now this time we have two calculations to do.

For the first one, £19 is the *old* amount; the *new* amount is what we are trying to find, and the *factor* will be 0.88 this time (can you have a guess why?).

So, from

$$new = old \times factor$$

we can write

$$new = 19 \times 0.88 = 16.72$$

For the second calculation, £26 is the *old* amount; the *new* amount is what we are trying to find, and the *factor* will be 0.88 again.

So, from

$$new = old \times factor$$

we can write

$$new = 26 \times 0.88 = 22.88$$

So the total amount we would have to pay for both items would be £16.72 + £22.88 = £39.60.

4 A Digression: The Factor

OK, let's have a look at the factor, and how you can get the factor from the percentage in the problem (and indeed vice versa, as we will see shortly). By the way, for those of you who are interested, Appendix A outlines *why* the factor is what it is.

For the percentage increase questions we have had these percentages, and associated factors:

Percentage Increase	**Factor**
5%	1.05
8%	1.08
15%	1.15

Can you see how to get the factor from the percentage? If not, here are a few more pairs - see if you can see the connection:

Percentage Increase	**Factor**
5%	1.05
8%	1.08
15%	1.15
7%	1.07
11%	1.11
25%	1.25

Hopefully, by now you might have twigged that you get the factor from the percentage this way: start with a factor of 1.00, then for a percentage *increase*, you have to *add* 0.01 to the factor for each percent in the problem.

Now let's look at our percentage decrease questions:

Percentage Decrease	**Factor**
3%	0.97
12%	0.88

This time you get the factor from the percentage this way: start with a factor of 1.00, then for a percentage *decrease*, you have to *subtract* 0.01 from the factor for each percent in the problem.

Here are a few more examples, to see if you get the idea.

Percentage Decrease	**Factor**
3%	0.97
12%	0.88
4%	0.96
11%	0.89
7%	0.93
6%	0.94
25%	0.75
18%	0.82

Get it?

Now, those of you who are on the ball might have twigged that as well as going from a percentage to a factor, we can go the other way - from the factor to the percentage. And whether the factor is more than one or less than one will determine if we have an increase or a decrease. For example, if you know that the factor is 1.09, you know that that must correspond to a 9% *increase*. And if you had a factor of 0.98, then that must correspond to a 2% *decrease*.

This is going to be very useful in the next section...

5 Reversed Percentages

Now we come to the bit where this method is so useful. Reverse percentage questions are very often tricky things that are hard to get your head around. However, using our scheme, they are no harder than the percentage increase and decrease questions we've already seen.

Here's an example.

? Question 6

Tina's pay is increased by 5% to £315. What was her pay before the increase?

And here's how I would answer this question.

Answer 6

As usual, we have to figure out what is going to be the *new* value, what is going to be the *old* value, and what the *factor* will be.

In this question, the £315 is going to be the *new* value, because that's her *new* wage. The *factor* will be 1.05, because it represents a 5% *increase*. And in this, as in all reversed percentage questions, it is the *old* value that we are trying to find.

So, from

$$\textit{new} = \textit{old} \times \textit{factor}$$

we can write

$$315 = \textit{old} \times 1.05$$

This time, we can't use the equation directly. Because it's the *old* value we want, we're going to have to rearrange the equation to make *old* the subject. We do this by dividing both sides by the *factor*:

$$\textit{old} = \frac{315}{1.05}$$

and now we can do the calculation:

$$\textit{old} = \frac{315}{1.05} = 300$$

So Tina's old wage was £300.

So actually, that was pretty straightforward. With reversed percentage questions, it's the *old* value we're trying to find, rather than the *new* value. But it's the same equation we use for reversed percentage questions that we used for percentage increase and decrease questions.

Here's another example.

Question 7

In a sale, a TV is reduced to a price of £325.50. This is a 7% reduction in the original price. What was the original price?

And here's my answer to this one.

Answer 7

As usual, we have to figure out what is going to be the *new* value, what is going to be the *old* value, and what the *factor* will be.

Here, the £325.50 is going to be the *new* value, because that's the *new* price of the TV. The *factor* will be 0.93, because it represents a 7% *decrease*. And in this, as in all reversed percentage questions, it is the *old* value that we are trying to find.

So, from

$$new = old \times factor$$

we can write

$$325.50 = old \times 0.93$$

As in the previous example, we can't use the equation directly. Because it's the *old* value we want, we're going to have to rearrange the equation to make *old* the subject. We do this by dividing both sides by the *factor*:

$$old = \frac{325.50}{0.93}$$

and now we can do the calculation:

$$old = \frac{325.50}{0.93} = 350$$

So the original price of the TV was £350.

6 Compound Interest and Repeated Percentages

Compound interest is the name given to a particular way of calculating the interest that is earned on certain financial packages, like bank accounts. For most savings accounts, the interest accrued on an account is determined using compound interest.

It works like this. Let's say that a bank pays compound interest on an account at the rate of 6% per year[2].

And let's say that you start with an amount of £500 in your account on January 1st, 2000.

Right, so as the interest is added yearly, on the 31st December 2000, 6% of £500 is added to the account. Another way of saying that is that there will be a 6% increase in the balance of the account.

So on the 1st of January 2001, the balance of the account will be

$$\textit{new balance} \;=\; 500 \times 1.06 \;=\; \pounds 530$$

i.e. after 1 year, there will be $500 \times 1.06 = \pounds 530$ in the account.

Now the process repeats: as the interest is added yearly, on the 31st December 2001, 6% of £530 is added to the account. [This is *compound* bit: the interest in the second year is 6% of the new balance, not 6% of the original balance.] Another way of saying that is that there will be a 6% increase in the balance of the account.

So on the 1st of January 2002, the new balance will be

$$\textit{new balance} \;=\; 530 \times 1.06 \;=\; \pounds 561.80$$

i.e. after 2 years, there will be $500 \times 1.06 \times 1.06 = \pounds 561.80$ in the account.

And that could be written as: after 2 years, there will be $500 \times 1.06^2 = \pounds 561.80$ in the account.

And for the third year the process repeats again: as the interest is added yearly, on the 31st December 2002, 6% of £561.80 is added to the account.

So on the 1st of January 2003, the new balance will be

$$\textit{new balance} \;=\; 561.80 \times \;1.06 \;=\; \pounds 595.51 \;(\textit{to the nearest penny})$$

i.e. after 3 years, there will be $500 \times 1.06 \times 1.06 \times 1.06 = \pounds 595.51$ in the account.

And that could be written as: after 3 years, there will be $500 \times 1.06^3 = \pounds 595.51$ in the account.

Can you see that we can generalise this to: after n years, there will be 500×1.06^n in the account?

This *factor* thing comes in really handy in these questions!

Repeated Percentage Formula

The formula for calculating *repeated* percentages is

$$new \;=\; old \;\times\; factor^n \tag{2}$$

where n is the number of years[a] over which the percentage increase or decrease is being applied.

[a]Or whatever time periods are relevant to the question.

Let's have a look at some examples to see how this works.

[2]Most accounts these days accumulate compound interest *daily*, but for the purposes of GCSE Maths questions, the interest is usually added *yearly*!

Question 8

Scientists have been studying the shores of Scotland and estimate that due to pollution, the seal population of those shores will decline at the rate of 15% each year. In 2006, they counted around 3000 seals on those shores.

(a) If nothing is done about pollution, how many seals will they expect to be there in (i) 2007? (ii) 2008? (iii) 2011?

(b) How long will it take for the seal population to be less than 1000?

And here's my answer to this one.

Answer 8

In these questions, we have to figure out what is going to be the *new* value, what is going to be the *old* value, and what the *factor* will be; but we also have to figure out what the n is going to be, too.

(i) Here, 3000 is going to be the *old* value, because that's the *old* population of seals. The *factor* will be 0.85, because that represents a 15% *decrease*. And it is the *new* value that we are trying to find. n here will just be 1, because the percentage decrease is so much *per year* and we want to know the number of seals *after 1 year*.

So, from

$$new \;=\; old \;\times\; factor^n$$

we can write

$$new \;=\; 3000 \;\times\; 0.85^1 \;=\; 2550$$

So the number of seals expected in 2007 is 2550.

(ii) This time, n here will be 2, because the percentage decrease is so much *per year* and we want to know the number of seals *after 2 years*.

$$new \;=\; 3000 \;\times\; 0.85^2 \;=\; 2167.5$$

So the number of seals expected in 2008 is about 2168 (you can't have half a seal!).

(iii) This time, n here will be 5, because we want to know the number of seals after 5 years.

$$new \;=\; 3000 \;\times\; 0.85^5 \;=\; 1331.1$$

So the number of seals expected in 2011 is about 1331.

(b) The easiest way to do this is to just use our formula to keep working out how many seals will be left after a given number of years (by changing the value of n):

Year	**Remaining Seals**
2011	1331
2012	1131
2013	962

So the number of seals expected in 2013 will be less than 1000.

Another question about repeated percentages.

? Question 9

The manager of a small family business offered his staff an annual pay increase of 4% for every year they stayed with the firm.

(a) Gareth started work at the firm on a salary of £12,200. What salary will he be on after 4 years?

(b) Julie started work at the business on a salary of £9,350. How many years will it be until she is earning a salary of over £20,000?

And here's my solution to this one.

Answer 9

In these questions, we have to figure out what is going to be the *new* value, what is going to be the *old* value, and what the *factor* will be; but we also have to figure out what the n is going to be, too.

(a) Here, 12200 is going to be the *old* value, because that's Gareth's *old* salary. The *factor* will be 1.04, because that represents a 4% *increase*. And it is the *new* value that we are trying to find. n here will be 4, because the percentage increase is so much *per year* and we want to know the salary after 4 years.

So, from

$$new \ = \ old \ \times \ factor^n$$

we can write

$$new \ = \ 12200 \ \times \ 1.04^4 \ = \ £14,272.27$$

So Gareth's salary after 4 years is £14,272.27.

(b) The easiest way to do this is to just use our formula to keep working out what Julie's salary will be after a given number of years (by changing the value of n):

Years (n)	**Salary**
1	£9,724.00
2	£10,112.96
3	£10,517.48

This is taking some time! I'm jumping to a much larger value of n!

Years (n)	**Salary**
15	£16,838.82
16	£17,512.37
17	£18,212.87
18	£18,941.38
19	£19,699.04
20	£20,487.00

So it will take 20 years for Julie to get a salary of over £20,000. I hope she enjoys the job!

7 One Quantity as a Percentage of Another

Now unfortunately, there is an exception to our simple rule of

$$new = old \times factor$$

And this is it! When you want to find "one quantity as a percentage of another", you do something slightly different. Here's how to do these:

Question 10

What is £5 as a percentage of £20?

Here's my answer:

Answer 10

Before I answer this, you might be able to guess that since £5 is a quarter of £20, then the answer should be 25%. So how do we get 25% from £5 and £20?

Well,

$$\frac{5}{20} \times 100 = 25$$

and that's how we do this type of question. Make a fraction of the two numbers, and multiply by 100 to turn it into a percentage.

Here's another example:

Question 11

What is £35 as a percentage of £125?

Here's my answer:

Answer 11

Well,

$$\frac{35}{125} \times 100 = 28$$

so £35 is 28% of £125.

Part II

If You Don't Have a Calculator

Occasionally you get a percentage question in the Non-Calculator paper. I know: percentages are bad enough as it is, never mind having to do them without a calculator.

The good news, though, is that you only get the easier type of questions in a Non-Calculator paper, and the numbers are easier. Let me give you an example. Here's a question:

? Question 12

What is 10% of 150?

And here's the answer:

Answer 12

In order to be able to do this without a calculator, there must be an easy way to work out 10% of something. And indeed there is!

It turns out that if you want to find 10% of something, you just divide the something by 10.

So, to find 10% of 150, we do this:

$$10\% \text{ of } 150 = \frac{150}{10} = 15$$

Oh, and I'm hoping that you know how to divide a number by 10. If you don't, go away NOW, and find out!

So, I'm hoping now that you can find 10% of something. Just divide the something by 10 to get the answer[3].

OK, so what if you want to find 20% of something? What do you do then? Well, if you know how to find 10% of something, to find 20% we just find 10% of it and double the answer. And that's because 20% is 2×10%!

And 30% is 3×10%, and so on.

What about 5%? Well, 5% will be half of 10% won't it?

[3]Note that this is NOT a general rule! For example, if you want to find 20% of something, you definitely do NOT divide the something by 20!! And if you want to find 5% of something, you definitely do NOT divide the something by 5!! It only works for 10%: it's a coincidence that the 10 is the same in this case.

Here's a couple of examples.

Question 13

What is 40% of 150?

And here's the answer:

Answer 13

To find 10% of 150, we do this:

$$10\% \text{ of } 150 = \frac{150}{10} = 15$$

And so to find 40% of 150, we do this:

$$40\% \text{ of } 150 = 4 \times 15 = 60$$

And finally:

Question 14

What is 5% of 180?

And here's the answer:

Answer 14

To find 10% of 180, we do this:

$$10\% \text{ of } 180 = \frac{180}{10} = 18$$

And so to find 5% of 180, we do this:

$$5\% \text{ of } 180 = \frac{18}{2} = 9$$

A Where the Factor Comes From

To see where the factor comes from, I'm going to run through a percentage increase question, and a percentage decrease question, showing all the steps from start to finish. Hopefully you will see the part played by the *factor* and where it comes from.

A.1 Percentage Increase

Here is a simple problem:

? Question 15

Increase £6.00 by 7%.

And here's my solution to this one.

Answer 15

If we want to increase £6.00 by 7%, what we'll need to do is first find what 7% of £6.00 is, and then add that to the original £6.00. OK, we calculate 7% of £6.00 using:

$$\frac{7}{100} \times 6.00$$

and then we have to add that on to the original £6.00:

$$6.00 \; + \; \frac{7}{100} \times 6.00$$

Now because we have two terms with something in common (the 6.00), we can *factorise* the 6.00 out:

$$6.00 \; \left\{ 1 \; + \; \frac{7}{100} \right\}$$

But now, if we write $\frac{7}{100}$ as a decimal (0.07), we get

$$6.00 \; \{1 \; + \; 0.07\}$$

which of course is equal to

$$6.00 \times 1.07$$

And lo! Our *factor* for a percentage increase appears!

A.2 Percentage Decrease

Now let's try the same thing, but with a decrease, instead of an increase.

? Question 16

Decrease £6.00 by 7%.

And here's my solution to this one.

Answer 16

If we want to decrease £6.00 by 7%, what we'll need to do is first find what 7% of £6.00 is, and then subtract that from the original £6.00. OK, we calculate 7% of £6.00 using:

$$\frac{7}{100} \times 6.00$$

and then we have to subtract that from the original £6.00:

$$6.00 \; - \; \frac{7}{100} \times 6.00$$

Now because we have two terms with something in common (the 6.00), we can *factorise* the 6.00 out:

$$6.00 \; \left\{1 \; - \; \frac{7}{100}\right\}$$

But now, if we write $\frac{7}{100}$ as a decimal (0.07), we get

$$6.00 \; \{1 \; - \; 0.07\}$$

which of course is equal to

$$6.00 \times 0.93$$

And lo! Our *factor* for a percentage decrease appears!