Steve Smith Tuition: Maths Notes

Integration by Substitution

Contents

Prerequisites

Knowledge of *the chain rule* and *implicit differentiation* for differentiating things would be useful.

Notes

None.

Document History

Date	Version	Comments
7th January 2016	1.0	Initial creation of the document.
11th January 2016	1.1	Updates...
6th March 2016	1.2	Modifying the special case!

1 Introduction

1.1 Integration as a Whole

You can only integrate what I call *standard forms*. These are functions that you just know the integrals of. They are primarily things like: polynomial terms ax^n; trigonometrical functions like $\sin(x)$ and $\cos(x)$; and other assorted things like e^x and $\ln(x)$. See Appendix A for a more extensive list of standard forms.

Quite often, you are given a formula sheet to take into an exam on integration, and in the formula sheet will be a list of functions and their integrals. In that case, everything in the list will be a standard form, as if you needed to integrate a function in the list, you could just write down the answer. There's no working to do.

So, what happens when you get an integral that's *not* a standard form?

Well, there's only *four* things you can do[1]. Each one of these four techniques is used to transform your integral into a standard form. The four techniques are:

- partial fractions;
- use of trigonometrical identities;
- substitution;
- integration by parts.

That's all.

Only those four. Nothing else.

In this document we will be looking at integration by substitution. I discuss integration by parts in Smith (2012a) and Smith (2012b).

1.2 Integration by Substitution

So, what is integration by substitution? We've already seen that it's a technique that transforms an integral you can't integrate into a standard form, but how does it do it?

Substitution is a *change of variable* technique. You may have come across a change of variable technique in differentiation. It's called *the chain rule*. Well, integration by substitution works in a similar way.

Let's see how...

[1] I'm talking here about *algaebraic* integration. You could always use *numerical* integration to integrate *any* function, even standard forms. That's a subject for another day...

2 The Basic Idea

Let me show you a simple example to take you through the idea. Let's say that we needed to work this out:

$$I = \int_{x=0}^{x=\frac{\pi}{6}} \cos(3x)\ dx \tag{1}$$

OK, well that looks simple enough. cos() is a standard form, after all: we could just go to Appendix A and look it up. It's...oh, hang on. $\cos(x)$ is a standard form. $\cos(3x)$ isn't!! Rats. So what do we do?

2.1 Transforming the Integrand

Since we know how to integrate $\cos(x)$, why don't we transform our integral into essentially that. And we could do that by introducing a new variable, u, say, like this:

$$u = 3x \tag{2}$$

so that our integral becomes

$$I = \int_{x=0}^{x=\frac{\pi}{6}} \cos(u)\ dx \tag{3}$$

Hey, hey! We now have something we *can* integrate. But whoa - there's a couple of problems here: now the function to be integrated, $\cos(u)$, has a different variable to the *limits* (the $x = 0$ and $x = \frac{\pi}{6}$ stuff), and the *differential* (the dx bit).

So not only are we going to have to transform the function to be integrated (known as the *integrand*), we are also going to have to transform the limits and the differential.

2.2 Transforming the Limits

Transforming the limits is pretty easy. What we need to do here is to find the value of u when $x = 0$ (the bottom limit), and the value of u when $x = \frac{\pi}{6}$ (the top limit). And how are we going to do that? Using the substitution (Equation (2)) of course: when $x = 0$, $u = 3 \times 0 = 0$, and when $x = \frac{\pi}{6}$, $u = 3 \times \frac{\pi}{6} = \frac{\pi}{2}$. So our integral now becomes:

$$I = \int_{u=0}^{u=\frac{\pi}{2}} \cos(u)\ dx \tag{4}$$

One interesting thing about integrals with limits is that *we don't always need to transform the limits*! I'll tell you more about this later. Obviously, if an integral doesn't have limits, we don't need to worry about this bit!

2.3 Transforming the Differential

Now there's only one thing left. We have to transform our differential dx into du. And how are we going to do that? Using the substitution (Equation (2)) of course! This time, though, there's a bit of work to do. As we need a relationshoip between dx and du, we will have to *differentiate* the substitution:

$$\begin{aligned} u &= 3x \\ \Rightarrow\ \frac{du}{dx} &= 3 \\ \Rightarrow\ dx &= \tfrac{1}{3}du \end{aligned} \tag{5}$$

We will always have to do this. So when I'm doing an integration by substitution, I differentiate my substitution as the first step of the whole process. Anyway, our integral now looks like this:

$$I = \int_{u=0}^{u=\frac{\pi}{2}} \cos(u)\ \tfrac{1}{3} du$$

which we could write as

$$I = \tfrac{1}{3} \int_{u=0}^{u=\frac{\pi}{2}} \cos(u)\ du \tag{6}$$

since $\frac{1}{3}$ is a *constant*, so we can take it outside the integral sign as the integration process won't affect it.

Now we have a standard form!! Yippee!! That was the whole point of this: $\cos(u)$ is in the list of standard forms[2].

So, now our integrand is a standard form, we can write down the integral as

$$I = \tfrac{1}{3} \Big[\sin(u) \Big]_{u=0}^{u=\frac{\pi}{2}}$$

and putting in the limits we get

$$\begin{aligned} I &= \tfrac{1}{3} \left\{ \left(\sin\left(\frac{\pi}{2}\right) \right) - \left(\sin(0) \right) \right\} \\ &= \tfrac{1}{3} \left\{ 1 - 0 \right\} \\ &= \tfrac{1}{3} \end{aligned}$$

2.4 Summary: The Basic Process

So, have a look at Figure 1. That gives you the process.

Start with an integral you don't know how to integrate

↓

Find a Substitution for your Variable

↓

Differentiate the Substitution

↓

Transform the Integrand

↓

Transform the Differential

↓

Transform the Limits (if necessary)

↓

Integrate!!

Figure 1: The Process

[2]The variable name, u, doesn't make any difference to this being a standard form. You will probably have the function $\cos(x)$ in your list of standard forms, integrating to $\sin(x)$. But of course that would mean that $\cos(u)$ would integrate to $\sin(u)$, $\cos(\eta)$ would integrate to $\sin(\eta)$, $\cos(*)$ would integrate to $\sin(*)$, etc. etc. The symbol used is irrelevant. Is that OK??

3 The Easiest Kind of Substitutions

The example in Section 2 is one of the easiest kinds of integrations by substitution you can get. This is all relative, of course: no integration by substitution is easy!!

What I mean by *the easiest kind of substitutions* is that after you have picked a substitution (more about this later!), and you have transformed the integrand and the differential, then the integral doesn't have any x stuff left in it, so we can integrate it straight away.

Here's another example. Let's integrate

$$I = \int x(5x^2 + 2)^3 \, dx$$

using the substitution $u = 5x^2 + 2$. Quite often, the substitution that you need to solve an integral will be given to you. Even in an exam! But somethimes it won't. What do you do then? Check out Section 8.

Anyway, here they are telling us what substitution to use, so let's use it. Remember that I said that we will always need to transform the differential by differentiating the substitution? Well, we will. So I always do it first.

$$\begin{aligned} u &= 5x^2 + 2 \\ \Rightarrow \frac{du}{dx} &= 10x \\ \Rightarrow dx &= \frac{1}{10x} du \end{aligned}$$

Right, so let's continue to follow the process: next up is to transform the integrand:

$$I = \int xu^3 \, dx$$

Clearly the $(5x^2 + 2)$ will transform to u, but what about the other x? Don't worry about this yet. Let's see what happens when we transform the differential:

$$I = \int xu^3 \frac{1}{10x} \, du$$

Now this is handy! The x we had left after transforming the integrand will cancel out when we transform the differential! Here goes:

$$\begin{aligned} I &= \int xu^3 \frac{1}{10x} \, du \\ &= \int u^3 \frac{1}{10} \, du \\ &= \tfrac{1}{10} \int u^3 \, du \end{aligned}$$

Cool! That has now transformed our integral into a standard form. And in this example we don't have any limits to transform, so we can get straight to the integration. This will now integrate to

$$\begin{aligned} I &= \tfrac{1}{10} \times \tfrac{1}{4} u^4 + C \\ &= \tfrac{1}{40} u^4 + C \end{aligned}$$

remembering the $+C$ when we don't have limits, of course. But we haven't finished yet! The original problem didn't have any us in it! It was specified in terms of x! So here, to finish off, we need to *substitute back* for the x:

$$\begin{aligned} I &= \tfrac{1}{40} u^4 + C \\ &= \tfrac{1}{40} \left(5x^2 + 2\right)^4 + C \end{aligned}$$

And now we're done!

4 The Hardest Kind of Substitutions

What I mean by *the hardest kind of substitutions* is that after you have picked a substitution and you have transformed the integrand and the differential, then the integral *does* have x stuff left in it, so we *can't* integrate it straight away.

Here's an example. Let's integrate

$$I = \int x\sqrt{2x+5}\, dx$$

using the substitution $u^2 = 2x + 5$.

Remember that I said that we will always need to transform the differential by differentiating the substitution? Well, we will. So I always do it first. Now because of the way that this substitution has been given to us, we can differentiate it *implicitly*:

$$\begin{aligned} u^2 &= 2x + 5 \\ \Rightarrow\ 2u\frac{du}{dx} &= 2 \\ \Rightarrow\ dx &= u\, du \end{aligned}$$

Right, so let's continue to follow the process: next up is to transform the integrand:

$$I = \int xu\, dx$$

Clearly the $\sqrt{2x+5}$ will transform to u, but what about the other x? Don't worry about this yet. Let's see what happens when we transform the differential:

$$\begin{aligned} I &= \int xuu\, du \\ &= \int xu^2\, du \end{aligned}$$

Now this isn't so good! The x we had left after transforming the integrand hasn't cancelled out when we transformed the differential! So what do we do? In this circumstance, we have to go back to the substitution, and see if we can make x the subject, so we get get the x in terms of u. This is the extra bit that makes this kind of integral harder than the easiest kind. But this isn't so bad! From our substitution,

$$x = \frac{u^2 - 5}{2}$$

and so if we plug this in to our integral, we get

$$\begin{aligned} I &= \int xu^2\, du \\ &= \int \frac{u^2-5}{2}u^2\, du \\ &= \tfrac{1}{2}\int (u^2-5)u^2\, du \\ &= \tfrac{1}{2}\int u^4 - 5u^2\, du \end{aligned}$$

And we have transformed our integral into two (this time) standard forms, which we can go on to integrate:

$$\begin{aligned} I &= \tfrac{1}{2}\left[\tfrac{1}{5}u^5 - \tfrac{5}{3}u^3\right] \\ &= \tfrac{1}{10}u^5 - \tfrac{5}{6}u^3 + C \end{aligned}$$

Again at this point we need to substitute back for the x:

$$I = \tfrac{1}{10}\left(2x+5\right)^{\frac{5}{2}} - \tfrac{5}{6}\left(2x+5\right)^{\frac{3}{2}} + C$$

since $u^5 = (u^2)^{\frac{5}{2}}$, and $u^3 = (u^2)^{\frac{3}{2}}$.

5 "A Rose by any Other Name..."

An interesting thing about this change-of-variable technique is that it has so many names. Some books call it "The Reverse Chain Rule", or "The Use of Standard Patterns", or (my own personal favourite) "Integration by Recognition"[3].

This is needlessly complicating a relatively simple idea. And the idea is: one really important technique in integration is to change the variable, transforming a function that you can't integrate into one or more standard forms that you can integrate.

5.1 The "Reverse Chain Rule"

Here's an example from a C4 Edexcel book in great circulation, under the heading "You can integrate some functions using the reverse of the chain rule". Integrate

$$I = \int e^{4x+1}\, dx$$

Using our technique, the first thing we have to do is to pick a substitution. I'm going to use $u = 4x + 1$. The reason for this choice (if you can't twig already!) will be explained in Section 8. Next, we differentiate the substitution to get an expression for dx:

$$\begin{aligned} u &= 4x + 1 \\ \Rightarrow\ \frac{du}{dx} &= 4 \\ \Rightarrow\ dx &= \tfrac{1}{4}du \end{aligned}$$

Then we transform the integrand:

$$I = \int e^{u}\, dx$$

and the differential:

$$\begin{aligned} I &= \int e^{u}\ \tfrac{1}{4}\ du \\ &= \tfrac{1}{4}\int e^{u}\ du \end{aligned}$$

so that we now have a standard form which we can integrate:

$$\begin{aligned} I &= \tfrac{1}{4}e^{u} + C \\ &= \tfrac{1}{4}e^{4x+1} + C \end{aligned}$$

Is that not just substitution, then?

[3] I get really upset when students of mine get taught a solution technique called "Integration by Recognition". It's the *recognition* bit I have the problem with. Does this imply that there are other situations that you can use *recognition* with?

How about this one: Find the area of a circle of radius r. Aha, you can say - I can solve this by *recognition*: I recognise that I have to use the formula $A = \pi r^2$!

Or this one: Differentiate $y = x^2 \sin(x)$. Aha, you can say - I can solve this by *recognition*: I recognise that I have to use the product rule for this...

Recognition? Nonsense.

5.2 The Use of "Standard Patterns"

Here's another example from the same book, in the section "You can use standard patterns to integrate some expressions". Integrate

$$I = \int \frac{2x}{x^2+1}\,dx$$

Using our technique, the first thing we have to do is to pick a substitution. I'm going to use $u = x^2 + 1$. The reason for this choice (if you can't twig already!) will be explained in Section 8. Next, we differentiate the substitution to get an expression for dx:

$$\begin{aligned} u &= x^2 + 1 \\ \Rightarrow \frac{du}{dx} &= 2x \\ \Rightarrow dx &= \frac{1}{2x}du \end{aligned}$$

Then we transform the integrand:

$$I = \int \frac{2x}{u}\,dx$$

and the differential:

$$\begin{aligned} I &= \int \frac{2x}{u}\,\frac{1}{2x}\,du \\ &= \int \frac{1}{u}\,du \end{aligned}$$

so that we now have a standard form which we can integrate:

$$\begin{aligned} I &= \ln(u) + C \\ &= \ln(x^2+1) + C \end{aligned}$$

Is that not just substitution, then?

5.3 Integration by "Recognition"

Here's an example from another book in widespread use for C4 Edexcel Maths courses. Integrate

$$I = \int k(ax+b)^n\,dx$$

where a, b and k are all constants.

"Aha", you can say. "I can solve this by *recognition*: I recognise that I can use *substitution*!"

Following our technique, the first thing to do is to find a substitution. I'm going to choose $u = ax + b$ (see Section 8). Next thing - differentiate the substitution to obtain an expression for dx:

$$\begin{aligned} u &= ax + b \\ \Rightarrow \frac{du}{dx} &= a \\ \Rightarrow dx &= \frac{1}{a}\,du \end{aligned}$$

Then we transform the integrand:

$$I = \int ku^n\,dx$$

and the differential:

$$\begin{aligned} I &= \int ku^n \, \frac{1}{a} \, du \\ &= \frac{k}{a} \int u^n \, du \end{aligned}$$

so that we now have a standard form which we can integrate:

$$\begin{aligned} I &= \frac{k}{a} \, \frac{1}{n+1} u^{n+1} + C \\ &= \frac{k}{a(n+1)} (ax+b)^{n+1} + C \end{aligned}$$

Is that not just substitution, then?

In the book, it says that to integrate things like this you should *recognise* that the integral is the differential of something. You then have to differentiate your guess and compare it with the integral you have. Now there is a place for this sort of thing: guessing a solution and working backwards. But the place for it is usually when you can't think of anything better to do. But we *can* think of something better to do. It's called *substitution*!!

6 A Word About Limits

I hinted earlier that you might not always need to transform the limits of an integral when you use substitution. Here's an example of what I mean. Let's say that you want to find this integral:

$$I = \int_{x=0}^{x=1} (x+5)^2 \, dx$$

And I'm going to find this integral two ways: one where I transform the limits, and one where I don't.

6.1 Transforming the Limits

OK, this is just proceeding as before. First, find a substitution. This one is easy to spot: $u = x + 5$. Next, differentiate the substitution:

$$\begin{aligned} u &= x + 5 \\ \Rightarrow \; \frac{du}{dx} &= 1 \\ \Rightarrow \; dx &= du \end{aligned}$$

Then I transform the integrand:

$$I = \int_{x=0}^{x=1} u^2 \, dx$$

and the differential

$$I = \int_{x=0}^{x=1} u^2 \, du$$

and the limits. Here, when $x = 0$, $u = 5$, and when $x = 1$, $u = 6$. So

$$I = \int_{u=5}^{u=6} u^2 \, du$$

Now I can integrate

$$\begin{aligned} I &= \left[\tfrac{1}{3}u^3\right]_{u=5}^{u=6} \\ &= \left\{ \left(\tfrac{1}{3}6^3\right) - \left(\tfrac{1}{3}5^3\right) \right\} \\ &= \left\{ \frac{216}{3} - \frac{125}{3} \right\} \\ &= \frac{91}{3} \end{aligned}$$

6.2 Without Transforming the Limits

It's possible to find the value of this integral *without* transforming the limits! Do do this, we proceed as per the last section until we have transformed the integrand and the differential (but *not* the limits):

$$I = \int_{x=0}^{x=1} u^2 \, du$$

Now everything *inside* the integral sign is u stuff, so we could integrate it:

$$I = \left[\tfrac{1}{3}u^3\right]_{x=0}^{x=1}$$

But hang on a minute, how do we substitute in for values of x? Oh. of course! We could switch back to x stuff inside the square brackets:

$$I = \left[\tfrac{1}{3}(x+5)^3\right]_{x=0}^{x=1}$$

Now we can shove the numbers in:

$$\begin{aligned} I &= \left[\tfrac{1}{3}(x+5)^3\right]_{x=0}^{x=1} \\ &= \left\{ \left(\tfrac{1}{3}(1+5)^3\right) - \left(\tfrac{1}{3}(0+5)^3\right) \right\} \\ &= \left\{ \left(\tfrac{1}{3}6^3\right) - \left(\tfrac{1}{3}5^3\right) \right\} \\ &= \left\{ \frac{216}{3} - \frac{125}{3} \right\} \\ &= \frac{91}{3} \end{aligned}$$

as before.

So: you always have a choice as to whether you transform the limits, or transform the u integral back to x stuff. And there's no golden rule here: just do the one that's easiest. Or the one you're most comfortable with.

6.3 Good Practice with Limits

One of the mistakes I used to make when I used integration by substitution with limits was to forget to change the limits. That's easily done if you write an integral like this:

$$I = \int_0^1 (x-1)^2 \, dx$$

Now if you use the substitution $u = x - 1$, then $dx = du$, and the integral transforms into

$$I = \int_0^1 u^2 \, du$$

right? Wrong!! We've forgotten to change the limits. And that's really easy to do if you don't explicitly include the variable in the limits.

So, get into the habit of:

$$\text{using} \qquad \int_{x=0}^{x=1} (x-1)^2 \, dx \qquad \text{instead of} \qquad \int_0^1 (x-1)^2 \, dx$$

It will pay dividends in the long run, I promise you!

7 A Very Special Case

There is a special case that you should be intimately familiar with. This is...

$$I = \int f'(x) \cdot g\Big[f(x)\Big] \, dx$$

Know this. It comes up a lot. Here are some examples...

7.1 Integrating $\frac{f'(x)}{f(x)}$

Now this is a very special type of integral indeed. Integrate

$$I = \int \frac{f'(x)}{f(x)} \, dx \tag{7}$$

This is the special case, with $g(x) = \frac{1}{x}$.

Now this is completely general: we can have *any function we like* on the bottom of the fraction, so long as we have the derivative of that function on the top. So - how do we integrate this? Well, you can use substitution, and the substitution is $u = f(x)$. So, the first thing we have to do is to differentiate the substitution:

$$\begin{aligned} u &= f(x) \\ \Rightarrow \frac{du}{dx} &= f'(x) \\ \Rightarrow dx &= \frac{1}{f'(x)} du \end{aligned}$$

OK, now we transform the integrand

$$I = \int \frac{f'(x)}{u} \, dx$$

and the differential

$$\begin{aligned} I &= \int \frac{f'(x)}{u} \, \frac{1}{f'(x)} \, du \\ &= \int \frac{1}{u} \, du \end{aligned}$$

Blimey! This is just one of our standard forms, so

$$\begin{aligned} I &= \ln\Big[u\Big] + C \\ &= \ln\Big[f(x)\Big] + C \end{aligned}$$

So, the first thing I do when I get an integral to solve is check to see if it's a fraction. And if it is, I check to see if the top is the differential of the bottom. And if it is, I can just write the answer down.

For example,

$$I = \int \frac{6x+5}{3x^2+5x-2}\,dx$$

would integrate to

$$I = \ln(3x^2+5x-2) + C$$

Or

$$I = \int \cot(x)\,dx = \int \frac{\cos(x)}{\sin(x)}\,dx$$

would integrate to

$$I = \ln(\sin(x)) + C$$

You can even use this idea if the top of the fraction in the integrand isn't quite the differential of the bottom, but it is a multiple of it. For example, with

$$I = \int \frac{12x+10}{3x^2+5x-2}\,dx$$

we could start by writing this as

$$\begin{aligned} I &= \int \frac{2(6x+5)}{3x^2+5x-2}\,dx \\ &= 2\int \frac{6x+5}{3x^2+5x-2}\,dx \end{aligned}$$

so it would integrate to

$$I = 2\ln(3x^2+5x-2) + C$$

Or if we had this integral

$$I = \int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx$$

we could start by writing this as

$$I = -\int \frac{-\sin(x)}{\cos(x)}\,dx$$

by inserting a negative sign both inside and outside of the integral (so they cancel each other out). The purpose, of course, of this is to ensure that we now have the differential of the bottom on the top. So,

$$\begin{aligned} I &= -\ln\Big[\cos(x)\Big] + C \\ &= \ln\left(\Big[\cos(x)\Big]^{-1}\right) + C \\ &= \ln\Big[\sec(x)\Big] + C \end{aligned}$$

I can't emphasise just how important it is to know this. Integrals of this kind crop up on every exam paper. It's that important.

So, just to try and get across just how important this one is to remember, I've drawn a nice little eye-catching picture to try and get it deep within the little grey cells:

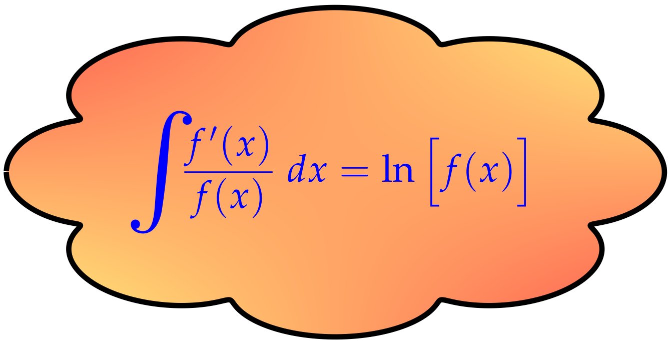

Figure 2: A Useful Thing To Know...

7.2 Integrating $f'(x) \cdot \left[f(x) \right]^n$

This is the special case, with $g(x) = x^n$.

Now this is another completely general integral: we can have *any function we like* as our $f(x)$, so long as we have the derivative of that function multiplying the power of it. So - how do we integrate this? Well, you can use substitution, and the substitution is $u = f(x)$. So, the first thing we have to do is to differentiate the substitution:

$$\begin{aligned} u &= f(x) \\ \Rightarrow \frac{du}{dx} &= f'(x) \\ \Rightarrow dx &= \frac{1}{f'(x)} du \end{aligned}$$

OK, now we transform the integrand

$$I = \int f'(x) \cdot u^n \, dx$$

and the differential

$$\begin{aligned} I &= \int f'(x) \cdot u^n \cdot \frac{1}{f'(x)} \, du \\ &= \int u^n \, du \end{aligned}$$

Blimey! This is just one of our standard forms, so

$$\begin{aligned} I &= \frac{1}{n+1} u^{n+1} + C \\ &= \frac{1}{n+1} \left[f(x) \right]^{n+1} + C \end{aligned}$$

You don't have to remember this result. But *do* remember that you use substitution to solve it.

7.3 Integrating $f'(x) \cdot e^{f(x)}$

This is the special case, with $g(x) = e^x$.

And again this is another completely general integral: we can have *any function we like* as our $f(x)$, so long as we have the derivative of that function multiplying the exponential of it. So - how do we integrate this? Well, you can use substitution, and the substitution is $u = f(x)$. So, the first thing we have to do is to differentiate the substitution:

$$\begin{aligned} u &= f(x) \\ \Rightarrow \frac{du}{dx} &= f'(x) \\ \Rightarrow dx &= \frac{1}{f'(x)} du \end{aligned}$$

OK, now we transform the integrand

$$I = \int f'(x) \cdot e^u \; dx$$

and the differential

$$\begin{aligned} I &= \int f'(x) \cdot e^u \cdot \frac{1}{f'(x)} \; du \\ &= \int e^u \; du \end{aligned}$$

Blimey! This is just one of our standard forms, so

$$\begin{aligned} I &= e^u + C \\ &= e^{f(x)} + C \end{aligned}$$

You don't have to remember this result. But *do* remember that you use substitution to solve it.

7.4 Integrating $f'(x) \cdot g[f(x)]$

So...the most general special case possible: we can have *any function we like* as our $g(x)$, and we can have *any function we like* as our $f(x)$, so long as we have the derivative of that function multiplying the $g[f(x)]$. So - how do we integrate this? Well, you can use substitution, and the substitution is $u = f(x)$. So, the first thing we have to do is to differentiate the substitution:

$$\begin{aligned} u &= f(x) \\ \Rightarrow \frac{du}{dx} &= f'(x) \\ \Rightarrow dx &= \frac{1}{f'(x)} du \end{aligned}$$

OK, now we transform the integrand

$$I = \int f'(x) \cdot g(u) \; dx$$

and the differential

$$\begin{aligned} I &= \int f'(x) \cdot g(u) \cdot \frac{1}{f'(x)} \; du \\ &= \int g(u) \; du \end{aligned}$$

And if we can integrate this, we are done. Are you getting the idea?

8 How to Come Up With a Substitution

If you are not given the substitution in the question, how do you go about finding it? This can be tricky. And you will not always get it right first time. But with a bit of practice, you will develop a bit of experience and can usually come up with the right substitution within a couple of goes. And if you don't get it right first time, it will usually become apparent quite quickly, so you won't have wasted a lot of time.

But: are there any guidelines that you can use to come up with a substitution? Well, remember the basic plan? What we are trying to do is to transform our integral into a standard form. So the first thing I do is to look at the standard forms and see which one my integral is closest to.

8.1 Which Standard Form is my Integral Closest To?

For example, if my integral is

$$I = \int \cos(3x)\ dx$$

then the integrand is obviously closest to $\cos(x)$, so the transformation $3x \to x$ is the one we want. This leads us to $u = 3x$.

As another example of this kind, if my integral is

$$I = \int \frac{1}{3x+2}\ dx$$

then the integrand is obviously closest to $\frac{1}{x}$, so the transformation $3x + 2 \to x$ is the one we want. This leads us to $u = 3x + 2$.

Here's another example. Find

$$I = \int e^{4x+1}\ dx$$

Well, the integrand is obviously closest to e^x, so the transformation $4x + 1 \to x$ is the one we want. This leads us to $u = 4x + 1$.

8.2 Letting *u* be the Trickiest Bit

Looking at the standard forms and seeing which one the integral is closest to doesn't always work. What about this:

$$I = \int x(5x^2 + 2)^3\ dx$$

My second guideline is: "What is the *trickiest* bit?" What would a simple substitution simplify *most*?

Here, I think the trickiest bit is the $(5x^2 + 2)^3$ bit. That would be a lot simpler if it was just x^3 (which is one of our standard forms). So maybe the transformation $5x^2 + 2 \to x$ is the one we want. That would lead to letting $u = 5x^2 + 2$.

In Section 4 we came across the integral

$$I = \int x\sqrt{2x+5}\ dx$$

Here I reckon that the trickiest bit is the $\sqrt{2x+5}$ bit. So, what if we used $u = \sqrt{2x+5}$? Well, let's try it. First we need to differentiate the substitution. That's slightly awkward (as we need the chain rule), but not too bad:

$$\begin{aligned} u &= \sqrt{2x+5} = (2x+5)^{\frac{1}{2}} \\ \Rightarrow \frac{du}{dx} &= \tfrac{1}{2}(2x+5)^{-\frac{1}{2}} \times 2 \\ \Rightarrow \frac{du}{dx} &= (2x+5)^{-\frac{1}{2}} \\ \Rightarrow \frac{du}{dx} &= \frac{1}{(2x+5)^{\frac{1}{2}}} \\ \Rightarrow \frac{du}{dx} &= \frac{1}{u} \\ \Rightarrow dx &= u\ du \end{aligned}$$

So after we've transformed the integrand and the differential we've got

$$\begin{aligned} I &= \int xuu\ du \\ &= \int xu^2\ du \end{aligned}$$

So what do we do about the extra x? Oh yes, we have to use the substitution to convert it to u stuff. Now since

$$\begin{aligned} u &= \sqrt{2x+5} \\ \Rightarrow u^2 &= 2x+5 \\ \Rightarrow x &= \frac{u^2-5}{2} \end{aligned}$$

and our integral becomes

$$= \int \frac{u^2-5}{2} u^2\ du$$

as it did in section 4.

But $u = \sqrt{2x+5}$ wasn't the substitution we used in Section 4! We used $u^2 = 2x+5$. And I chose that because it's easier to differentiate!!

What this means then is that it may well be possible to transform your integral via *more than one substitution*. Cool!

Here's another example. Find

$$I = \int \frac{2x}{x^2+1}\ dx$$

This is a fraction, and the only fraction standard form is $\frac{1}{x}$. So what about trying $u = x^2+1$? This works (see Section 5.2).

8.3 Finding a Substitution: Summary

So...my main guidelines so far are:

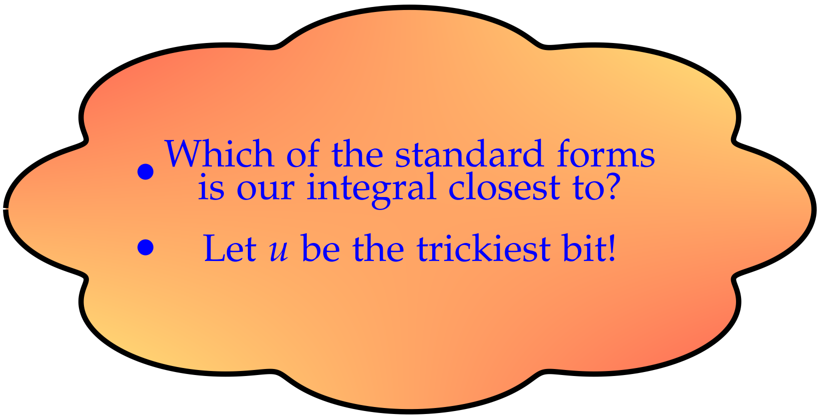

Figure 3: Finding a Substitution

9 Examples

9.1 Example 1

The Problem

Evaluate

$$I = \int_{x=2}^{x=5} \frac{1}{(x-1)^3}\, dx$$

The Solution Technique

Integration by substitution is a change of variable technique. We want to change the variable so that our integral turns from something we can't integrate into something we can integrate, what I would call *a standard form*. The only things that we can integrate are standard forms (see Appendix A).

When we change the variable in an integral, there are *three* things that we have to look at changing:

- the function to be integrated (called the *integrand*);
- the *differential* (the dx bit);
- and (possibly) the limits.

Choosing the Substitution

In this question we are not told the substitution. So, what can it be? The first thing that strikes me is that we have a fraction here. Now - is the top the differential of the bottom? No, nothing like. Rats. So we can't use the special case (see Section 7). So...

The integrand looks a bit like $\frac{1}{x^3}$, which is a standard form, so I'm going to try the substitution

$$u = x - 1$$

Differentiating the Substitution

Now remember that we will always have to differentiate the substitution to find an expression for dx in terms of du:

$$\begin{aligned} u &= x - 1 \\ \Rightarrow \frac{du}{dx} &= 1 \\ \Rightarrow dx &= du \end{aligned}$$

Changing the Integrand

Using our substitution to change the integrand, our integral becomes

$$I = \int_{x=2}^{x=5} \frac{1}{u^3}\, dx$$

Changing the Differential

When we change the differential our integral becomes

$$I = \int_{x=2}^{x=5} \frac{1}{u^3}\, du$$

Changing the Limits

In this question, there are limits, so we have a choice of how to handle them. Let's change the limits this time. So, using our substitution, when $x = 2$, $u = 1$, and when $x = 5$, $u = 4$. So, after changing the limits, our integral becomes

$$I = \int_{u=1}^{u=4} \frac{1}{u^3}\, du$$

Doing the Integration

Finally we have achieved a standard form. We can now integrate it:

$$\begin{aligned}
I &= \int_{u=1}^{u=4} \frac{1}{u^3}\, du \\
&= \int_{u=1}^{u=4} u^{-3}\, du \\
&= \Big[-\tfrac{1}{2}u^{-2}\Big]_{u=1}^{u=4} \\
&= \left[-\frac{1}{2u^2}\right]_{u=1}^{u=4} \\
&= \left[\left(-\frac{1}{2\times 4^2}\right) - \left(-\frac{1}{2\times 1^2}\right)\right] \\
&= \left[\left(-\frac{1}{32}\right) - \left(-\frac{1}{2}\right)\right] \\
&= \frac{1}{2} - \frac{1}{32} \\
&= \frac{16}{32} - \frac{1}{32} \\
&= \frac{15}{32}
\end{aligned}$$

9.2 Example 2

The Problem

Evaluate

$$I = \int_{x=4}^{x=6} \frac{x}{x-2}\, dx$$

The Solution Technique

Integration by substitution is a change of variable technique. We want to change the variable so that our integral turns from something we can't integrate into something we can integrate, what I would call *a standard form*. The only things that we can integrate are standard forms (see Appendix A).

When we change the variable in an integral, there are *three* things that we have to look at changing:

- the function to be integrated (called the *integrand*);
- the *differential* (the dx bit);
- and (possibly) the limits.

Choosing the Substitution

In this question we are not told the substitution. So, what can it be? The first thing that strikes me is that we have a fraction here. Now - is the top the differential of the bottom? No, it's not. Rats. So we can't use the special case (see Section 7). So...

The integrand looks a bit like $\frac{1}{x}$, which is a standard form, so I'm going to try the substitution

$$u = x - 2$$

Differentiating the Substitution

Now remember that we will always have to differentiate the substitution to find an expression for dx in terms of du:

$$\begin{aligned} u &= x - 2 \\ \Rightarrow \frac{du}{dx} &= 1 \\ \Rightarrow dx &= du \end{aligned}$$

Changing the Integrand

Using our substitution to change the integrand, our integral becomes

$$I = \int_{x=4}^{x=6} \frac{x}{u}\, dx$$

Changing the Differential

When we change the differential our integral becomes

$$I = \int_{x=4}^{x=6} \frac{x}{u}\, du$$

Now we still have an x in our integrand. So, we go back to the substitution, and get x in terms of u. Using the substitution, $x = u + 2$, so our integral becomes

$$I = \int_{x=4}^{x=6} \frac{u+2}{u}\, du$$

which we can write (due to the way fractions add) as

$$\begin{aligned} I &= \int_{x=4}^{x=6} \frac{u}{u} + \frac{2}{u}\, du \\ &= \int_{x=4}^{x=6} 1 + \frac{2}{u}\, du \\ &= \int_{x=4}^{x=6} 1\, du \;+\; 2\int_{x=4}^{x=6} \frac{1}{u}\, du \end{aligned}$$

Changing the Limits

In this question, there are limits, so we have a choice of how to handle them. Let's *not* change the limits this time!

Doing the Integration

Finally we have achieved a standard form. We can now integrate it:

$$\begin{aligned} I &= \int_{x=4}^{x=6} 1\, du \;+\; 2\int_{x=4}^{x=6} \frac{1}{u}\, du \\ &= \Big[u\Big]_{x=4}^{x=6} \;+\; 2\Big[\ln(u)\Big]_{x=4}^{x=6} \end{aligned}$$

At this point, because the limits are limits of the wrong variable, we have to substitute the x stuff back in:

$$\begin{aligned} I &= \Big[x-2\Big]_{x=4}^{x=6} \;+\; 2\Big[\ln(x-2)\Big]_{x=4}^{x=6} \\ &= \Big[4-2\Big]_{x=4}^{x=6} \;+\; 2\Big[\ln(4)-\ln(2)\Big]_{x=4}^{x=6} \\ &= 2 + 2\ln(2) \end{aligned}$$

using the fact that $\ln(4) - \ln(2) = \ln\left(\frac{4}{2}\right)$, so

$$\begin{aligned} I &= 2 + 2\ln(2) \\ &= 2 + \ln(2^2) \\ &= 2 + \ln(4) \end{aligned}$$

9.3 Example 3

The Problem

Find

$$I = \int \frac{3x}{x^2 - 1}\, dx$$

The Solution Technique

Integration by substitution is a change of variable technique. We want to change the variable so that our integral turns from something we can't integrate into something we can integrate, what I would call *a standard form*. The only things that we can integrate are standard forms (see Appendix A).

When we change the variable in an integral, there are *three* things that we have to look at changing:

- the function to be integrated (called the *integrand*);
- the *differential* (the dx bit);
- and (possibly) the limits.

It's the Special Case!

In this question we are not told the substitution. So, what can it be? The first thing that strikes me is that we have a fraction here. Now - is the top the differential of the bottom? Not quite, but it's close! If it was $2x$ instead of $3x$ it would be! Aha! In that case, I'm going to change my integral like this

$$\begin{aligned} I &= \tfrac{3}{2} \int \frac{\frac{2}{3} \times 3x}{x^2 - 1}\, dx \\ &= \tfrac{3}{2} \int \frac{2x}{x^2 - 1}\, dx \end{aligned}$$

Now the top of my integrand *is* the differential of the bottom. So I can just write the answer down (see Section 7):

$$I = \tfrac{3}{2} \ln\left(x^2 - 1\right) + C$$

9.4 Example 4

The Problem

Integrate

$$I = \int \frac{x}{\sqrt{x-2}}\,dx$$

using the substitution

$$u = \sqrt{x-2}$$

The Solution Technique

Integration by substitution is a change of variable technique. We want to change the variable so that our integral turns from something we can't integrate into something we can integrate, what I would call *a standard form*. The only things that we can integrate are standard forms (see Appendix A).

When we change the variable in an integral, there are *three* things that we have to look at changing:

- the function to be integrated (called the *integrand*);
- the *differential* (the dx bit);
- and (possibly) the limits.

Choosing the Substitution

In this question we are told the substitution.

Differentiating the Substitution

Usually, the way to transform the differential from dx into du is to *differentiate* the substitution. This gives us a way to write dx in terms of du. So let's do that first:

$$\begin{aligned} u &= \sqrt{x-2} = (x-2)^{\frac{1}{2}} \\ \Rightarrow \frac{du}{dx} &= \tfrac{1}{2}(x-2)^{-\frac{1}{2}} \times 1 \qquad \text{(using the chain rule)} \\ \Rightarrow \frac{du}{dx} &= \tfrac{1}{2}(x-2)^{-\frac{1}{2}} \end{aligned}$$

So to find out what dx would turn into, we make it the subject of this formula by multiplying both sides by dx, 2, and $(x-2)^{\frac{1}{2}}$:

$$2(x-2)^{\frac{1}{2}}\,du = dx$$

But $(x-2)^{\frac{1}{2}} = u$! So:

$$2u\,du = dx$$

Interestingly, we could have got this result from the substitution in a different way. Since $u = \sqrt{x-2}$, then $u^2 = x-2$. Now we can differentiate this *implicitly* to yield:

$$\begin{aligned} u^2 &= x-2 \\ \Rightarrow 2u\frac{du}{dx} &= 1 \\ \Rightarrow 2u\,du &= dx \end{aligned}$$

which is neater, easier and quicker!

Changing the Integrand

Using the substitution our integral becomes

$$I = \int \frac{x}{u}\, dx$$

Changing the Differential

Next, substitute for the differential, dx:

$$\begin{aligned} I &= \int \frac{x}{u}\, 2u\, du \\ &= \int 2x\, du \end{aligned}$$

Rats! The x is still there: the only thing that cancelled was the u. So what are we going to do with the x? Ah - hang on, earlier we used the substitution to show that $u^2 = x - 2$. So in that case, $x = u^2 + 2$! This is what you have to do if you've still got an x left after you've changed the integrand and the differential. So,

$$\begin{aligned} I &= \int 2x\, du \\ &= \int 2(u^2 + 2)\, du \\ &= \int 2u^2 + 4\, du \end{aligned}$$

And this is something that you should be able to integrate, because it is a combination of standard forms (we just use the polynomial rule to integrate this).

Changing the Limits

In this question, there are no limits, so we don't have to worry about changing those.

Doing the Integration

So, finally,

$$\begin{aligned} I &= \int 2u^2 + 4\, du \\ &= \tfrac{2}{3}u^3 + 4u + C \end{aligned}$$

And now all that's left is to put the x stuff back in:

$$I = \tfrac{2}{3}(x-2)^{\frac{3}{2}} + 4(x-2)^{\frac{1}{2}} + C$$

Now if you want to be really flash, you could massage this a bit:

$$\begin{aligned} I &= (x-2)^{\frac{1}{2}} \left[\tfrac{2}{3}(x-2) + 4\right] + C \\ &= (x-2)^{\frac{1}{2}} \left[\tfrac{2}{3}x - \tfrac{4}{3} + \tfrac{12}{3}\right] + C \\ &= (x-2)^{\frac{1}{2}} \left[\tfrac{2}{3}x + \tfrac{8}{3}\right] + C \\ &= \tfrac{2}{3}(x-2)^{\frac{1}{2}} \left[x + 4\right] + C \end{aligned}$$

9.5 Example 5

The Problem

Using the substitution $u = x - 1$, integrate

$$I = \int (x+2)(x-1)^4 \, dx$$

The Solution Technique

Integration by substitution is a change of variable technique. We want to change the variable so that our integral turns from something we can't integrate into something we can integrate, what I would call *a standard form*. The only things that we can integrate are standard forms (see Appendix A).

When we change the variable in an integral, there are *three* things that we have to look at changing:

- the function to be integrated (called the *integrand*);
- the *differential* (the dx bit);
- and (possibly) the limits.

Choosing the Substitution

In this question we are told the substitution.

Differentiating the Substitution

Usually, the way to transform the differential from dx into du is to *differentiate* the substitution. This gives us a way to write dx in terms of du. So let's do that first:

$$\begin{aligned} u &= x - 1 \\ \Rightarrow \frac{du}{dx} &= 1 \end{aligned}$$

So to find out what dx would turn into, we make it the subject of this formula by multiplying both sides by dx:

$$du = dx$$

Changing the Integrand

Using our substitution, our integral becomes

$$I = \int (x+2)u^4 \, dx$$

Changing the Differential

Substituting for the differential, our integral becomes

$$I = \int (x+2)u^4 \, du$$

Rats! The x is still there. So what are we going to do with the x? We can use the substitution to show that $x = u + 1$! This is what you have to do if you've still got an x left after you've changed the integrand and the differential. So,

$$\begin{aligned} I &= \int (x+2)u^4 \, du \\ &= \int (u+1+2)u^4 \, du \\ &= \int (u+3)u^4 \, du \\ &= \int u^5 + 3u^4 \, du \end{aligned}$$

And this is something that you should be able to integrate, because it is a combination of standard forms (we just use the polynomial rule to integrate it).

Changing the Limits

In this question, there are no limits, so we don't have to worry about changing those.

Doing the Integration

So, finally,

$$\begin{aligned} I &= \int u^5 + 3u^4 \, du \\ &= \tfrac{1}{6}u^6 + \tfrac{3}{5}u^5 + C \\ &= u^5 \left[\tfrac{1}{6}u + \tfrac{3}{5}\right] + C \end{aligned}$$

And now all that's left is to put the x stuff back in:

$$I = (x-1)^5 \left[\tfrac{1}{6}(x-1) + \tfrac{3}{5}\right] + C$$

Now if you want to be really flash, you could massage this a bit:

$$\begin{aligned} I &= (x-1)^5 \left[\tfrac{1}{6}(x-1) + \tfrac{3}{5}\right] + C \\ &= (x-1)^5 \left[\tfrac{1}{6}x - \tfrac{1}{6} + \tfrac{3}{5}\right] + C \\ &= (x-1)^5 \left[\tfrac{5}{30}x - \tfrac{5}{30} + \tfrac{18}{30}\right] + C \\ &= (x-1)^5 \left[\tfrac{5}{30}x + \tfrac{13}{30}\right] + C \\ &= \tfrac{1}{30}(x-1)^5 \left[5x + 13\right] + C \end{aligned}$$

A Standard Forms

Here is a list of standard forms. It's not a comprehensive list, but it includes some important integrals that you either have to know (the top half of the table), or will be in your formula booklet (the bottom half).

Function	Integral
ax^n	$\frac{a}{n+1}x^{n+1}$
$\sin(x)$	$-\cos(x)$
$\cos(x)$	$\sin(x)$
e^x	e^x
$\ln(x)$	$\frac{1}{x}$
a^{bx}	$\frac{a^{bx}}{b\ln(a)}$
$\sec^2(kx)$	$\frac{1}{k}\tan(kx)$
$\sec(x)\ \tan(x)$	$\sec(x)$
$\tan(x)$	$\ln(\lvert\sec(x)\rvert)$
$\cot(x)$	$\ln(\lvert\sin(x)\rvert)$
$\text{cosec}(x)$	$-\ln(\lvert\text{cosec}(x)+\cot(x)\rvert)$
$\sec(x)$	$\ln(\lvert\sec(x)+\tan(x)\rvert)$
$-\text{cosec}^2(x)$	$\cot(x)$
$-\text{cosec}(x)\ \cot(x)$	$\text{cosec}(x)$

Table 1: Integral Standard Forms

References

Smith, S. (2012a). Maths Notes : Integration by Parts, the Tabular Method I: "DIS is how you do it!". How to do integration by parts the easy way.

Smith, S. (2012b). Maths Notes : Integration by Parts, the Tabular Method II: "DIS is how you do more with it!". How to do even more integrations by parts the easy way.