Steve Smith Tuition: Maths Notes

How To Do Integration II: The Questions

Contents

Prerequisites

Knowledge of the four techniques of integral transformation is required!!

Notes

There is an accompanying document to this one: Smith (2016). There you will find an outline of how to tackle A2-Level integration questions. Read that before looking at this one.

Document History

Date	Version	Comments
6^{th} March 2016	1.0	Initial creation of the document.

1 Edexcel C4 June 2005

Question 3

The Question

(a) Express

$$\frac{5x+3}{(2x-3)(x+2)}$$

in partial fractions. **(3 marks)**

(b) Hence find the exact value of

$$I = \int_{x=2}^{x=6} \frac{5x+3}{(2x-3)(x+2)}\, dx$$

giving your answer as a single logarithm. **(5 marks)**

My Answer

(a) There's a recipe for doing these. As this is a document about integration techniques, I'm not going to cover how to answer this type of thing here. I'm going to assume that you know how to do this. If you do, you should get the answer

$$\frac{5x+3}{(2x-3)(x+2)} = \frac{3}{2x-3} + \frac{1}{x+2}$$

(b) So our integral will become

$$\int_{x=2}^{x=6} \frac{3}{2x-3} + \frac{1}{x+2}\, dx = \int_{x=2}^{x=6} \frac{3}{2x-3}\, dx + \int_{x=2}^{x=6} \frac{1}{x+2}\, dx \tag{1}$$

separating out each term into its own integral. You don't have to do this of course, but sometimes it's convenient, as it is here. Just give me a minute and I'll get to why!

Now **all** partial fractions integrals end up with terms like this

$$\int \frac{a}{bx+c}\, dx$$

to integrate. And there is a standard way that you do these. When you learn about using substitution to solve integrals, there is a very common family of integrals that you learn about. These integrals are

$$\int f'(x) \cdot [f(x)]^n\, dx, \qquad \int f'(x) \cdot e^{f(x)}\, dx, \qquad \int \frac{f'(x)}{[f(x)]^n}\, dx, \qquad \text{indeed...} \qquad \int f'(x) \cdot g[f(x)]\, dx$$

where the coloured bits are the things to look out for. If you have an integral of one of these types, then you use substitution to transform the integral into a standard form, and the substitution is $u = f(x)$.

So for me, the very first thing that I do when I get an integral is to ask myself "Is it a fraction?" And if the answer is *yes*, I then ask "Is the top the differential of the bottom?" Because if the answer to that question is *yes* as well, then I can just write the answer down. I'll show you what I mean. Take the first integral in our problem (1) here:

$$I_1 = \int_{x=2}^{x=6} \frac{3}{2x-3}\, dx$$

Now this *is* a fraction! And the top is the differential of the bottom! Well, not quite, but *I can make it be the differential of the bottom*:

$$I_1 = \int_{x=2}^{x=6} \frac{3}{2x-3}\, dx = \frac{3}{2} \int_{x=2}^{x=6} \frac{2}{2x-3}\, dx$$

See what I've done here? I've made the numerator in the fraction 2 (which is the differential of the bottom), and multiplied the whole integral by $\frac{3}{2}$ to compensate! Cunning! Now we have the pattern for a substitution. And the substitution is $u = 2x-3$:

$$\text{If} \qquad u = 2x-3 \qquad \text{then} \qquad \frac{du}{dx} = 2 \qquad \text{and} \qquad dx = \tfrac{1}{2}\, du$$

Now we can transform our variable form x to u:

$$I_1 = \frac{3}{2}\int_{x=2}^{x=6} \frac{2}{u}\cdot\frac{1}{2}\,du$$
$$= \frac{3}{2}\int_{x=2}^{x=6} \frac{1}{u}\,du$$

This is now a *standard form*. So we can just write down the integral:

$$I_1 = \tfrac{3}{2}\Big[\ln(u)\Big]_{x=2}^{x=6}$$

Now I can shove the x stuff back in so that we can evaluate this thing (which I will do in a minute):

$$I_1 = \tfrac{3}{2}\Big[\ln(2x-3)\Big]_{x=2}^{x=6}$$

Notice that

$$\int \frac{f'(x)}{f(x)}\,dx = \ln\left[f(x)\right]$$

So, let's have a look at the second integral in (1). This is

$$I_2 = \int_{x=2}^{x=6} \frac{1}{x+2}\,dx$$

Aha! This is a fraction, where the top is the differential of the bottom! So I can just *write the answer down* (or go through the substitution of $u = x+2$ if you really must!) to give:

$$I_2 = \int_{x=2}^{x=6} \frac{1}{x+2}\,dx = \Big[\ln(x+2)\Big]_{x=2}^{x=6}$$

Putting this altogether, then,

$$\begin{aligned} I &= \tfrac{3}{2}\Big[\ln(2x-3)\Big]_{x=2}^{x=6} + \Big[\ln(x+2)\Big]_{x=2}^{x=6} \\ &= \tfrac{3}{2}\Big[\ln(9)-\ln(1)\Big] + \Big[\ln(8)-\ln(4)\Big] \\ &= \tfrac{3}{2}\Big[\ln(9)\Big] + \Big[\ln(2)\Big] \\ &= \ln(27)+\ln(2) \\ &= \ln(54) \end{aligned}$$

Tips, Tricks and Patterns

Patterns:

$$\int f'(x)\cdot\left[f(x)\right]^n\,dx, \qquad \int f'(x)\cdot e^{f(x)}\,dx, \qquad \int \frac{f'(x)}{\left[f(x)\right]^n}\,dx \qquad \text{indeed...} \qquad \int f'(x)\cdot g\left[f(x)\right]\,dx$$

Question 4

The Question

Use the substitution $x = \sin(\theta)$ to find the exact value of

$$I = \int_{x=0}^{x=\frac{1}{2}} \frac{1}{\left[1-x^2\right]^{\frac{3}{2}}}\,dx$$

(7 marks)

My Answer

Well, this is an easy one! Easy, that is, from the point of view of which technique to use! We are told to use substitution, and also which substitution to use! So let's get going.

$$\text{If} \qquad x = \sin(\theta) \qquad \text{then} \qquad \frac{dx}{d\theta} = \cos(\theta) \qquad \text{and} \qquad dx = \cos(\theta)\ d\theta$$

Leaving the limits for a moment, let's transform the integrand (the function to be integrated) and the differential. I becomes:

$$I = \int_{x=0}^{x=\frac{1}{2}} \frac{1}{\left[1-\sin^2(\theta)\right]^{\frac{3}{2}}}\ \cos(\theta)\ d\theta$$

OK, so how does that help? Well, I'm hoping that you spot something in the denominator of this integral. Specifically, the $1 - \sin^2(\theta)$ bit. Recognise this? Ah - we could use $\sin^2(\theta) + \cos^2(\theta) = 1$ to simplify this:

$$I = \int_{x=0}^{x=\frac{1}{2}} \frac{1}{\left[\cos^2(\theta)\right]^{\frac{3}{2}}}\ \cos(\theta)\ d\theta$$

Now the powers in the denominator can be combined using the index rules:

$$\begin{aligned} I &= \int_{x=0}^{x=\frac{1}{2}} \frac{1}{\cos^3(\theta)}\ \cos(\theta)\ d\theta \\ &= \int_{x=0}^{x=\frac{1}{2}} \frac{1}{\cos^2(\theta)}\ d\theta \\ &= \int_{x=0}^{x=\frac{1}{2}} \sec^2(\theta)\ d\theta \end{aligned}$$

Now looking in the formula book (in the C4 integration table) we find that $\sec^2(x)$ integrates to $\tan(x)$!! Yippee!!! We now have a standard form then that we can integrate:

$$I = \Big[\tan(x)\Big]_{x=0}^{x=\frac{1}{2}}$$

Now we have the limits to worry about. From the original substitution $x = \sin(x)$, then when $x = 0$ then $\theta = 0°$, and when $x = \frac{1}{2}$ then $\theta = \sin^{-1}(\frac{1}{2}) = 30°$, so:

$$I = \Big[\tan(x)\Big]_{\theta=0°}^{\theta=30°}$$

or

$$I = \Big[\tan(30°) - \tan(0°)\Big]$$

or

$$I = \left[\frac{1}{\sqrt{3}} - 0\right]$$

so

$$I = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Tips, Tricks and Patterns

Tricks:

- $\sin^2(\theta) + \cos^2(\theta) \equiv 1$

Question 5

The Question

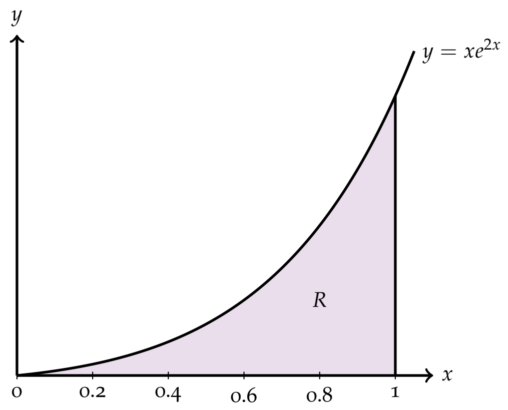

Figure 1: A graph of $y = xe^{2x}$

Figure 1 shows the graph of the curve with the equation

$$y = xe^{2x} \qquad\qquad x \geq 0$$

The finite region R bounded by the lines $x = 1$, the x-axis and the curve is shown shaded in Figure 1.

(a) Use integration to find the exact value of the area for R. **(5 marks)**

(b) ...

(c) ...

My Answer

We need to find

$$\int_{x=0}^{x=1} xe^{2x}\ dx$$

Well, this is a classic integration by parts. Whenever we have a polynomial multiplied by a sine, cosine, exponential or log, it's parts. So...DIS is how we do it:

D	I	S
x	e^{2x}	$+$
1	$\frac{1}{2}e^{2x}$	$-$
0	$\frac{1}{4}e^{2x}$	$+$
		$-$

Figure 2: Integrating $\int xe^{2x}\ dx$

So that

$$\begin{aligned}\int_{x=0}^{x=1} xe^{2x}\ dx &= \left[\tfrac{1}{2}xe^{2x} - \tfrac{1}{4}e^{2x}\right]_{x=0}^{x=1} \\ &= \left[\left\{\tfrac{1}{2}\cdot 1\cdot e^{2} - \tfrac{1}{4}e^{2}\right\} - \left\{\tfrac{1}{2}\cdot 0\cdot e^{0} - \tfrac{1}{4}e^{0}\right\}\right] \\ &= \left[\left\{\tfrac{1}{4}e^{2}\right\} - \left\{-\tfrac{1}{4}\right\}\right] \\ &= \tfrac{1}{4}\left[e^{2} + 1\right]\end{aligned}$$

Question 8

The Question

Liquid is pouring into a container at a constant rate of 20 cm^3 s^{-1} and is leaking out at a rate proportional to the volume of the liquid already in the container.

(a) Explain why, at time t seconds, the volume, V cm^3, of liquid in the container satisfies the differential equation

$$\frac{dV}{dt} = 20 - kV$$

where k is a positive constant. **(2 marks)**

The container is initially empty.

(b) By solving the differential equation, show that

$$V = A + Be^{-kt}$$

giving the values of A and B in terms of k. **(6 marks)**

Given also that $\frac{dV}{dt} = 10$ when $t = 5$,

(c) find the volume of liquid in the container at 10 s after the start. **(5 marks)**

My Answer

(b) We have to solve the differential equation

$$\frac{dV}{dt} = 20 - kV$$

At A-level, there is only one way we can solve differential equations: we have to *separate the variables*:

$$\frac{1}{20 - kV}\, dV = dt$$

and then integrate both sides

$$\int \frac{1}{20 - kV}\, dV = \int dt$$

The right hand side of this is easy: that's just going to integrate to t. But what about the left-hand side? Well, that's a fraction! Hello - is the top the differential of the bottom? Yes! Well, almost! But we can make the top the differential of the bottom:

$$-\frac{1}{k} \int \frac{-k}{20 - kV}\, dV = \int dt$$

To do that I had to have a $-k$ on the top. That means I would have to multiply the integral by $-k$. So to compensate, I divided the integral by $-k$ as well! Now the left hand side is one of our patterns: if we use substitution ($u = 20 - kV$) then we will end up with:

$$-\frac{1}{k} \ln(20 - kV) + \ln(C) = t$$

using $\ln(C)$ as the integration constant. Multiplying both sides by $-k$ gives

$$\ln(20 - kV) - k \ln(C) = -kt$$

or

$$\ln(20 - kV) - \ln(D) = -kt$$

where I have absorbed the k into the C to form a new constant, and so

$$\ln\left(\frac{20 - kV}{D}\right) = -kt$$

Raising each side as a power of e gives

$$\frac{20 - kV}{D} = e^{-kt}$$

so that

$$20 - kV = De^{-kt}$$

and that

$$kV = 20 - De^{-kt}$$

and finally

$$V = \left(\frac{20}{k}\right) - \left(\frac{D}{k}\right) e^{-kt}$$

which is of the required form. Now we use the initial condition, *the container is initially empty*, that is: $V = 0$ when $t = 0$.

$$0 = \left(\frac{20}{k}\right) - \left(\frac{D}{k}\right) e^{0}$$

So $D = 20$, and our formula becomes

$$V = \left(\frac{20}{k}\right) - \left(\frac{20}{k}\right) e^{-kt} = \left(\frac{20}{k}\right) \left[1 - e^{-kt}\right]$$

(c) Since we now know V as a function of t, then from the differential equation,

$$\begin{aligned} \frac{dV}{dt} &= 20 - kV = 20 - k\left(\frac{20}{k}\right)\left[1 - e^{-kt}\right] \\ &= 20 - 20\left[1 - e^{-kt}\right] \\ &= 20 - 20 + 20e^{-kt} \\ &= 20e^{-kt} \end{aligned}$$

Now we can use the information: $\frac{dV}{dt} = 10$ when $t = 5$, so

$$10 = 20e^{-5k}$$

That means that

$$\tfrac{1}{2} = e^{-5k}$$

and so

$$\ln\left(\tfrac{1}{2}\right) = -5k$$

and

$$k = -\tfrac{1}{5}\ln\left(\tfrac{1}{2}\right)$$

And so, to find the volume of liquid in the tank when $t = 10$,

$$\begin{aligned} V &= \left(\frac{20}{k}\right)\left[1 - e^{-kt}\right] \\ &= \left(\frac{20}{-\frac{1}{5}\ln\left(\frac{1}{2}\right)}\right)\left[1 - e^{\frac{1}{5}\ln\left(\frac{1}{2}\right)\cdot 10}\right] \\ &= -\frac{100}{\ln\left(\frac{1}{2}\right)}\left[1 - e^{2\ln\left(\frac{1}{2}\right)}\right] \\ &= -\frac{100}{\ln\left(\frac{1}{2}\right)}\left[1 - e^{\ln\left(\frac{1}{4}\right)}\right] \\ &= -\frac{100}{\ln\left(\frac{1}{2}\right)}\left[1 - \tfrac{1}{4}\right] \\ &= -\frac{75}{\ln\left(\frac{1}{2}\right)} \\ &= \frac{75}{\ln(2)} \end{aligned}$$

Phew!

Tips, Tricks and Patterns

Patterns:

$$\int f'(x) \cdot [f(x)]^n \, dx, \qquad \int f'(x) \cdot e^{f(x)} \, dx, \qquad \int \frac{f'(x)}{[f(x)]^n} \, dx \qquad \text{indeed...} \qquad \int f'(x) \cdot g[f(x)] \, dx$$

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Question 3

The Question

Using the substitution $u^2 = 2x - 1$, or otherwise, find the exact value of

$$I = \int_{x=1}^{x=5} \frac{3x}{\sqrt{2x-1}}\, dx$$

(8 marks)

My Answer

Well, they're telling us it's substitution, and what substitution to use. So, here goes:

$$\text{If} \qquad u^2 = 2x - 1 \qquad \text{then} \qquad 2u\frac{du}{dx} = 2 \qquad \text{and} \qquad dx = u\, du$$

using a cunning bit of implicit differentiation! So, plugging this stuff in:

$$\begin{aligned} I &= \int_{x=1}^{x=5} \frac{3x}{u}\, u\, du \\ &= \int_{x=1}^{x=5} 3x\, du \end{aligned}$$

Now what are we going to do about the remaining $3x$? Well, we just go back to the substitution. Since $u^2 = 2x - 1$, then $2x = u^2 + 1$ and so $3x = \frac{3}{2}(u^2 + 1)$:

$$\begin{aligned} I &= \int_{x=1}^{x=5} \tfrac{3}{2}(u^2 + 1)\, du \\ &= \tfrac{3}{2} \int_{x=1}^{x=5} u^2 + 1\, du \\ &= \tfrac{3}{2}\left[\tfrac{1}{3}u^3 + u\right]_{x=1}^{x=5} \end{aligned}$$

Now from the substitution $u^2 = 2x - 1$, when $x = 5$, $u = 3$, and when $x = 1$, $u = 1$. So

$$\begin{aligned} I &= \tfrac{3}{2}\left[\tfrac{1}{3}u^3 + u\right]_{u=1}^{u=3} \\ &= \tfrac{3}{2}\left[\left\{\tfrac{1}{3}3^3 + 3\right\} - \left\{\tfrac{1}{3}1^3 + 1\right\}\right] \\ &= \tfrac{3}{2}\left[\left\{9 + 3\right\} - \left\{\tfrac{1}{3} + 1\right\}\right] \\ &= \tfrac{3}{2}\left[11 - \tfrac{1}{3}\right] \\ &= \tfrac{3}{2}\left[\tfrac{32}{3}\right] \\ &= 16 \end{aligned}$$

Question 4

The Question

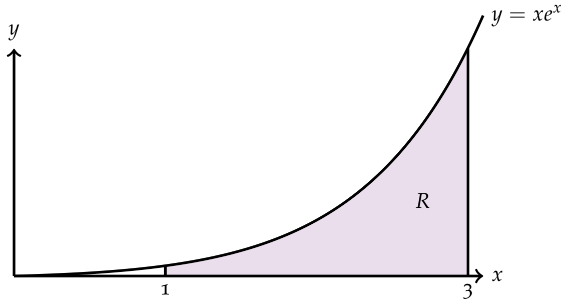

Figure 3: A graph of $y = xe^x$

Figure 3 shows the finite region R, which is bounded by the curve $y = xe^x$, the line $x = 1$, the line $x = 3$ and the x-axis.

The region R is rotated through 360° about the x-axis.

Use integration by parts to find an exact value for the **volume** of the solid generated. **(8 marks)**

My Answer

The integral we want is

$$\begin{aligned} V &= \pi \int y^2 \, dx \\ &= \pi \int_{x=1}^{x=3} x^2 e^{2x} \, dx \end{aligned}$$

Well, this is a classic integration by parts. Whenever we have a polynomial multiplied by a sine, cosine, exponential or log, it's parts. So...DIS is how we do it:

D	I	S
x^2	e^{2x}	$+$
$2x$	$\frac{1}{2}e^{2x}$	$-$
2	$\frac{1}{4}e^{2x}$	$+$
0	$\frac{1}{8}e^{2x}$	$-$
		$+$

Figure 4: Integrating $\int x^2 e^{2x} \, dx$

So, the integral we want is

$$\begin{aligned} V &= \pi \int_{x=1}^{x=3} x^2 e^{2x} \, dx \\ &= \pi \Big[\tfrac{1}{2}x^2 e^{2x} - \tfrac{1}{2}xe^{2x} + \tfrac{1}{4}e^{2x} \Big]_{x=1}^{x=3} \\ &= \pi \Big[\Big\{ \tfrac{3^2}{2}e^6 - \tfrac{1}{2}3e^6 + \tfrac{1}{4}e^6 \Big\} - \Big\{ \tfrac{1}{2}1^2e^2 - \tfrac{1}{2}1e^2 + \tfrac{1}{4}e^2 \Big\} \Big] \\ &= \pi \Big[\Big\{ \tfrac{18}{4}e^6 - \tfrac{6}{4}e^6 + \tfrac{1}{4}e^6 \Big\} - \Big\{ \tfrac{2}{4}e^2 - \tfrac{2}{4}e^2 + \tfrac{1}{4}e^2 \Big\} \Big] \\ &= \pi \Big[\tfrac{13}{4}e^6 - \tfrac{1}{4}e^2 \Big] \\ &= \frac{e^2 \pi}{4} \Big[13e^4 - 1 \Big] \end{aligned}$$

Question 7

The Question

(a) ...

(b) ...

(c) Given that $V = 0$ when $t = 0$, solve the differential equation

$$\frac{dV}{dt} = \frac{1000}{(2t+1)^2}$$

to obtain V in terms of t. **(4 marks)**

(d) ...

My Answer

To solve a differential equation we have to separate the variables,

$$dV = \frac{1000}{(2t+1)^2}\,dt$$

and integrate

$$\begin{aligned}\int dV &= \int \frac{1000}{(2t+1)^2}\,dt \\ &= 1000\int (2t+1)^{-2}\,dt\end{aligned}$$

The left hand side will simply integrate to V. What about the right hand side? This is not a standard form. Which one is it closest to? Well it looks like x^{-2} to me. That gives me the idea to try a substitution:

If $\quad u = 2t+1 \quad$ then $\quad \frac{du}{dt} = 2 \quad$ and $\quad dt = \frac{1}{2}\,du$

Throwing this stuff into the integral gives

$$\begin{aligned}\int dV &= 1000\int u^{-2}\,\tfrac{1}{2}\,du \\ &= 500\int u^{-2}\,du\end{aligned}$$

Fab. Let's integrate:

$$\begin{aligned}V &= -500u^{-1} + C \\ &= -500(2t+1)^{-1} + C \\ &= -\frac{500}{2t+1} + C\end{aligned}$$

Using the initial condition, $V = 0$ when $t = 0$ we get

$$0 = -\frac{500}{1} + C$$

so $C = 500$. Our formula for V is then

$$\begin{aligned}V &= 500 - \frac{500}{2t+1} \\ V &= 500\left[1 - \frac{1}{2t+1}\right]\end{aligned}$$

Tips, Tricks and Patterns

Tips:

- Which standard form is the integrand closest to?

Question 8

The Question

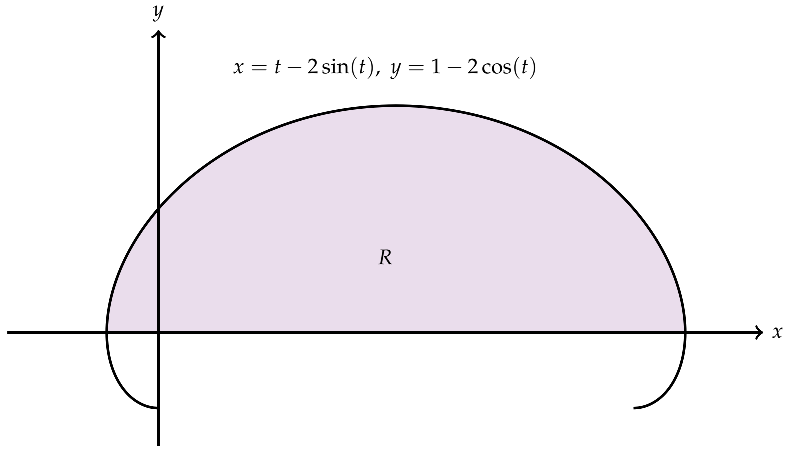

Figure 5: A graph of $x = t - 2\sin(t),\ y = 1 - 2\cos(t)$

The curve shown in Figure 5 has parametric equations

$$x = t - 2\sin(t), \qquad y = 1 - 2\cos(t), \qquad 0 \le t \le 2\pi$$

(a) Show that the curve crosses the x-axis where $t = \frac{\pi}{3}$ and $t = \frac{5\pi}{3}$. **(2 marks)**

The finite region R is enclosed by the curve and the x-axis, shown as shaded in Figure 5.

(b) Show that the area R is given by the integral

$$\int_{t=\frac{\pi}{3}}^{t=\frac{5\pi}{3}} [1 - 2\cos(t)]^2 \, dt$$

(3 marks)

(c) Use this integral to find the exact value of the shaded area. **(7 marks)**

My Answer

(a) As there are only 2 marks for this bit, all they need is for us to show that $x = 0$ when $t = \frac{\pi}{3}$ and when $t = \frac{5\pi}{3}$. Just plug these values into the calculator for the x-coordinate formula.

(b) So from part (a) the integral we want will be

$$I = \int_{t=\frac{\pi}{3}}^{t=\frac{5\pi}{3}} y \, dx$$

Now since $x = t - 2\sin(t)$, then $\frac{dx}{dt} = 1 - 2\cos(t)$, and so $dx = [1 - 2\cos(t)]\, dt$. So I becomes

$$I = \int_{t=\frac{\pi}{3}}^{t=\frac{5\pi}{3}} [1 - 2\cos(t)] \cdot [1 - 2\cos(t)] \, dt$$

$$= \int_{t=\frac{\pi}{3}}^{t=\frac{5\pi}{3}} [1 - 2\cos(t)]^2 \, dt$$

(c) When you have $\sin^2(x)$ or $\cos^2(x)$ to integrate, there is a trick. It's not easy to integrate powers of $\sin(x)$ and $\cos(x)$ higher than one, so we have to convert them to single powers using trigonometrical identities.

And the identities are: $\mathbf{\cos(2x) \equiv 2\cos^2(x) - 1 \equiv 1 - 2\sin^2(x)}$ (from the addition formulae). Using these identities we can convert either $\sin^2(x)$ or $\cos^2(x)$ into something with a $\cos(2x)$ in it.

How does this help here? Well, let's see what happens when we multiply out the brackets in our integral

$$I = \int_{t=\frac{\pi}{3}}^{t=\frac{5\pi}{3}} [1 - 2\cos(t)]^2 \, dt$$
$$= \int_{t=\frac{\pi}{3}}^{t=\frac{5\pi}{3}} 1 - 4\cos(t) + 4\cos^2(t) \, dt$$

But from $\cos(2x) \equiv 2\cos^2(x) - 1$, then $2\cos^2(x) \equiv \cos(2x) + 1$ and so $4\cos^2(x) \equiv 2\cos(2x) + 2$:

$$I = \int_{t=\frac{\pi}{3}}^{t=\frac{5\pi}{3}} 1 - 4\cos(t) + 2\cos(2t) + 2 \, dt$$
$$= \int_{t=\frac{\pi}{3}}^{t=\frac{5\pi}{3}} 3 - 4\cos(t) + 2\cos(2t) \, dt$$

Now everything in our integral is a standard form, so we can integrate this:

$$I = \Big[3t - 4\sin(t) + \sin(2t)\Big]_{t=\frac{\pi}{3}}^{t=\frac{5\pi}{3}}$$
$$= \left[\left\{3\frac{5\pi}{3} - 4\sin\left(\frac{5\pi}{3}\right) + \sin\left(2\frac{5\pi}{3}\right)\right\} - \left\{3\frac{\pi}{3} - 4\sin\left(\frac{\pi}{3}\right) + \sin\left(2\frac{\pi}{3}\right)\right\}\right]$$
$$= \left[\left\{5\pi + 2\sqrt{3} - \frac{\sqrt{3}}{2}\right\} - \left\{\pi - 2\sqrt{3} + \frac{\sqrt{3}}{2}\right\}\right]$$
$$= 4\pi + 3\sqrt{3}$$

Tips, Tricks and Patterns

Tricks:

When you have $\sin^2(x)$ or $\cos^2(x)$ to integrate, there is a trick.

You have to use the identities: $\mathbf{\cos(2x) \equiv 2\cos^2(x) - 1 \equiv 1 - 2\sin^2(x)}$.

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Question 3

The Question

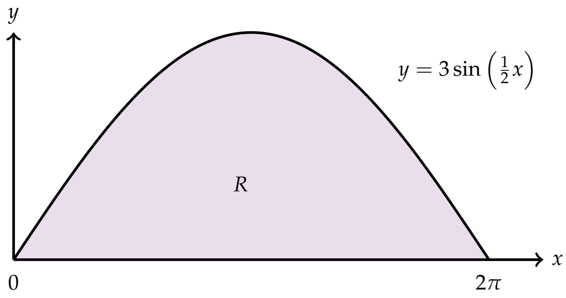

Figure 6: A graph of $y = 3\sin\left(\frac{1}{2}x\right)$

The curve with the equation $y = 3\sin\left(\frac{1}{2}x\right)$, $0 \leq x \leq 2\pi$, is shown in Figure 6. The finite region R enclosed by the curve and the x-axis is shaded.

(a) Find, by integration, the area of the shaded region R. **(3 marks)**

The region is rotated through 2π radians about the x-axis.

(b) Find the volume of the solid generated. **(6 marks)**

My Answer

(a) The integral we want is

$$I = \int_{x=0}^{x=2\pi} 3\sin\left(\tfrac{1}{2}x\right)\, dx$$

This is pretty close to being a standard form. The simple substitution $u = \frac{1}{2}x$ will turn it into one.

If $u = \frac{1}{2}x$ then $\frac{du}{dx} = \frac{1}{2}$ and $dx = 2\,du$

So...

$$\begin{aligned} I &= \int_{x=0}^{x=2\pi} 3\sin\left(\tfrac{1}{2}x\right)\, dx \\ &= 3\int_{x=0}^{x=2\pi} \sin(u)\, 2\, du \\ &= 6\int_{x=0}^{x=2\pi} \sin(u)\, du \end{aligned}$$

This is now a standard form, so we can integrate it:

$$\begin{aligned} I &= 6\Big[-\cos(u)\Big]_{x=0}^{x=2\pi} \\ &= 6\Big[-\cos\left(\tfrac{1}{2}x\right)\Big]_{x=0}^{x=2\pi} \\ &= 6\Big[\Big\{-\cos\left(\tfrac{1}{2}\cdot 2\pi\right)\Big\} - \Big\{-\cos\left(\tfrac{1}{2}\cdot 0\right)\Big\}\Big] \\ &= 6\Big[\Big\{1\Big\} - \Big\{-1\Big\}\Big] \\ &= 12 \end{aligned}$$

(b) The integral we want will be

$$\begin{aligned} V &= \pi \int_{x=0}^{x=2\pi} y^2 \, dx \\ &= \pi \int_{x=0}^{x=2\pi} \left[3\sin\left(\tfrac{1}{2}x\right)\right]^2 dx \\ &= 9\pi \int_{x=0}^{x=2\pi} \sin^2\left(\tfrac{1}{2}x\right) \, dx \end{aligned}$$

When you have $\sin^2(x)$ or $\cos^2(x)$ to integrate, there is a trick. It's not easy to integrate powers of $\sin(x)$ and $\cos(x)$ higher than one, so we have to convert them to single powers using trigonometrical identities.

And the identities are: $\mathbf{\cos(2x) \equiv 2\cos^2(x) - 1 \equiv 1 - 2\sin^2(x)}$ (from the addition formulae). Using these identities we can convert either $\sin^2(x)$ or $\cos^2(x)$ into something with a $\cos(2x)$ in it.

From these identities, $2\sin^2(x) \equiv 1 - \cos(2x)$, so $\sin^2\left(\frac{1}{2}x\right) \equiv \frac{1}{2} - \frac{1}{2}\cos(x)$. Throwing this into our integral gives

$$V = 9\pi \int_{x=0}^{x=2\pi} \tfrac{1}{2} - \tfrac{1}{2}\cos(x) \, dx$$

and now our integral consists of standard forms, so we can integrate it:

$$\begin{aligned} V &= 9\pi \left[\tfrac{1}{2}x - \tfrac{1}{2}\sin(x)\right]_{x=0}^{x=2\pi} \\ &= 9\pi \left[\left\{\tfrac{1}{2} \cdot 2\pi - \tfrac{1}{2}\sin(2\pi)\right\} - \left\{\tfrac{1}{2} \cdot 0 - \tfrac{1}{2}\sin(0)\right\}\right] \\ &= 9\pi \left[\left\{\pi - 0\right\} - \left\{0 - 0\right\}\right] \\ &= 9\pi^2 \end{aligned}$$

Tips, Tricks and Patterns

Tricks:

When you have $\sin^2(x)$ or $\cos^2(x)$ to integrate, there is a trick.

You have to use the identities: $\mathbf{\cos(2x) \equiv 2\cos^2(x) - 1 \equiv 1 - 2\sin^2(x)}$.

Question 6

The Question

(a) ...

(b) ...

(c) ...

(d) Show, by integration, that the exact value of

$$\int_{x=1}^{x=3} (x-1)\ln(x) \, dx$$

is $\frac{3}{2}\ln(3)$. **(6 marks)**

My Answer

(d) We need to find

$$I = \int_{x=1}^{x=3} (x-1)\ln(x) \, dx$$

D	I	S
$\ln(x)$	$x-1$	$+$
$\frac{1}{x}$	$\frac{1}{2}x^2 - x$	$-$
		$+$

Figure 7: Integrating $\int (x-1)\ln(x)\ dx$

Well, this is a classic integration by parts. Whenever we have a polynomial multiplied by a sine, cosine, exponential or log, it's parts. So...DIS is how we do it. As we don't know how to integrate $\ln(x)$, that has to go in the D column:

So our integral becomes

$$\begin{aligned} I &= \left[\left(\tfrac{1}{2}x^2 - x\right)\ln(x) \right]_{x=1}^{x=3} - \int_{x=1}^{x=3} \frac{1}{x} \cdot \left(\frac{1}{2}x^2 - x\right)\ dx \\ &= \left[\left(\tfrac{1}{2}x^2 - x\right)\ln(x) \right]_{x=1}^{x=3} - \int_{x=1}^{x=3} \frac{1}{2}x - 1\ dx \\ &= \left[\left(\tfrac{1}{2}x^2 - x\right)\ln(x) \right]_{x=1}^{x=3} - \left[\tfrac{1}{4}x^2 - x \right]_{x=1}^{x=3} \\ &= \left[\left(\tfrac{1}{2}x^2 - x\right)\ln(x) - \tfrac{1}{4}x^2 + x \right]_{x=1}^{x=3} \end{aligned}$$

Plugging the numbers in:

$$\begin{aligned} I = &= \left[\left\{ \left(\tfrac{1}{2} \cdot 3^2 - 3\right)\ln(3) - \tfrac{1}{4} \cdot 3^2 + 3 \right\} - \left\{ \left(\tfrac{1}{2} \cdot 1^2 - 1\right)\ln(1) - \tfrac{1}{4} \cdot 1^2 + 1 \right\} \right] \\ &= \left[\left\{ \tfrac{3}{2}\ln(3) + \tfrac{3}{4} \right\} - \left\{ 0 + \tfrac{3}{4} \right\} \right] \\ &= \tfrac{3}{2}\ln(3) \end{aligned}$$

Question 7

The Question

(a) ...

(b) ...

(c) Given that $V = 8$ when $t = 0$, solve the differential equation

$$\frac{dV}{dt} = 2V^{\frac{1}{3}}$$

and find the value of t when $V = 16\sqrt{2}$. **(7 marks)**

My Answer

(c) The only way we know how to solve differential equations is to separate the variables,

$$\frac{1}{V^{\frac{1}{3}}}\ dV = 2\ dt$$

and integrate:

$$\int \frac{1}{V^{\frac{1}{3}}}\ dV = \int 2\ dt$$

These are both standard forms!!!

$$\int V^{-\frac{1}{3}}\ dV = \int 2\ dt$$

and so

$$\tfrac{3}{2}V^{\frac{2}{3}} = 2t + C$$

Using the initial condition $V = 8$ when $t = 0$, then

$$\begin{aligned} \tfrac{3}{2} \cdot 8^{\frac{2}{3}} &= 2 \cdot 0 + C \\ \tfrac{3}{2} \cdot 4 &= C \\ 6 &= C \end{aligned}$$

and so

$$\begin{aligned} \tfrac{3}{2} V^{\frac{2}{3}} &= 2t + 6 \\ V^{\frac{2}{3}} &= \tfrac{2}{3}(2t + 6) \\ V^{\frac{2}{3}} &= \tfrac{4}{3}t + 4 \\ V &= \left[\tfrac{4}{3}t + 4\right]^{\frac{3}{2}} \end{aligned}$$

When $V = 16\sqrt{2}$,

$$\begin{aligned} 16\sqrt{2} &= \left[\tfrac{4}{3}t + 4\right]^{\frac{3}{2}} \\ 256 \cdot 2 &= \left[\tfrac{4}{3}t + 4\right]^{3} \\ 512 &= \left[\tfrac{4}{3}t + 4\right]^{3} \\ 8 &= \tfrac{4}{3}t + 4 \\ 4 &= \tfrac{4}{3}t \\ 3 &= t \end{aligned}$$

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Question 2

The Question

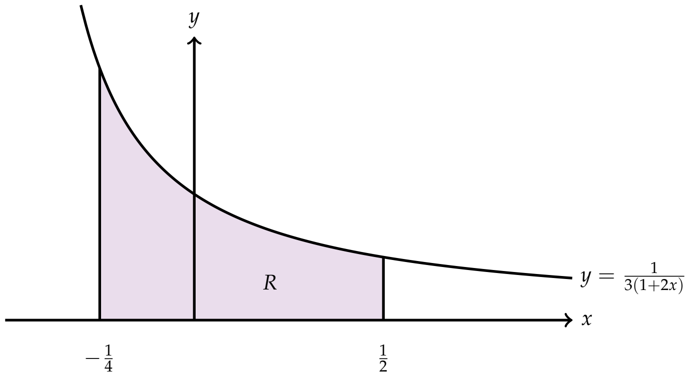

Figure 8: A graph of $y = \frac{1}{3(1+2x)}$

The curve with the equation

$$y = \frac{1}{3(1+2x)}, \qquad x > -\tfrac{1}{2}$$

is shown in Figure 8.

The region R, bounded by the lines $x = -\frac{1}{4}$, $x = \frac{1}{2}$, the x-axis and the curve, is shown shaded in Figure 8.

This region R is rotated through 360° about the x-axis.

(a) Use calculus to find the exact value of the volume of the solid generated. **(5 marks)**

(b) ...

My Answer

(a) The integral we want will be

$$\begin{aligned} I &= \pi \int_{x=-\frac{1}{4}}^{x=\frac{1}{2}} y^2 \, dx \\ &= \pi \int_{x=-\frac{1}{4}}^{x=\frac{1}{2}} \left[\frac{1}{3(1+2x)}\right]^2 dx \\ &= \frac{\pi}{9} \int_{x=-\frac{1}{4}}^{x=\frac{1}{2}} (1+2x)^{-2} \, dx \end{aligned}$$

This isn't a standard form. But it's close to one. It reminds me of x^{-2}. Ah! That gives me an idea for a substitution:

If $\quad u = 1 + 2x \quad$ then $\quad \frac{du}{dx} = 2 \quad$ and $\quad dx = \frac{1}{2} \, du$

Plugging this stuff into our integral gives

$$\begin{aligned} I &= \frac{\pi}{9} \int_{x=-\frac{1}{4}}^{x=\frac{1}{2}} u^{-2} \, \tfrac{1}{2} \, du \\ &= \frac{\pi}{18} \int_{x=-\frac{1}{4}}^{x=\frac{1}{2}} u^{-2} \, du \end{aligned}$$

This is now a standard form, so we can integrate it:

$$\begin{aligned} I &= \frac{\pi}{18}\left[-\frac{1}{u}\right]_{x=-\frac{1}{4}}^{x=\frac{1}{2}} \\ &= \frac{\pi}{18}\left[-\frac{1}{1+2x}\right]_{x=-\frac{1}{4}}^{x=\frac{1}{2}} \end{aligned}$$

Plugging the numbers in we get

$$\begin{aligned} I &= \frac{\pi}{18}\left[\left\{-\frac{1}{1+2\cdot\frac{1}{2}}\right\}-\left\{-\frac{1}{1+2\cdot-\frac{1}{4}}\right\}\right] \\ &= \frac{\pi}{18}\left[\left\{-\tfrac{1}{2}\right\}-\left\{-2\right\}\right] \\ &= \frac{\pi}{18}\cdot\frac{3}{2} \\ &= \frac{\pi}{12} \end{aligned}$$

Tips, Tricks and Patterns

Tips:

- What standard form is the integrand closest to?

Question 4

The Question

(a) Express

$$\frac{2x-1}{(x-1)(2x-3)}$$

in partial fractions. **(3 marks)**

(b) Given that $x \geq 2$, find the general solution of the differential equation

$$(x-1)(2x-3)\frac{dy}{dx} = (2x-1)y$$

(5 marks)

(c) Hence find the particular solution of this differential equation that satisfies $y = 10$ at $x = 2$, giving your answer in the form $y = f(x)$. **(4 marks)**

My Answer

(a) It turns out that

$$\frac{2x-1}{(x-1)(2x-3)} = \frac{4}{2x-3} - \frac{1}{x-1}$$

(b) The only way we can solve differential equations is to separate the variables,

$$\frac{1}{y}\ dy = \frac{2x-1}{(x-1)(2x-3)}\ dx$$

and integrate both sides

$$\int \frac{1}{y}\ dy = \int \frac{2x-1}{(x-1)(2x-3)}\ dx$$

which, following on from part (a), we can write as

$$\int \frac{1}{y}\ dy = \int \frac{4}{2x-3} - \frac{1}{x-1}\ dx$$

or indeed

$$\int \frac{1}{y}\ dy = \int \frac{4}{2x-3}\ dx - \int \frac{1}{x-1}\ dx$$

Now **all** partial fractions integrals end up with terms like this

$$\int \frac{a}{bx+c}\,dx$$

to integrate. And there is a standard way that you do these. When you learn about using substitution to solve integrals, there is a very common family of integrals that you learn about. These integrals are

$$\int f'(x)\cdot[f(x)]^n\,dx, \qquad \int f'(x)\cdot e^{f(x)}\,dx, \qquad \int \frac{f'(x)}{[f(x)]^n}\,dx, \qquad \text{indeed...} \qquad \int f'(x)\cdot g[f(x)]\,dx$$

where the coloured bits are the things to look out for. If you have an integral of one of these types, then you use substitution to transform the integral into a standard form, and the substitution is $u = f(x)$.

So, with this in mind, we can change our integral equation slightly to

$$\int \frac{1}{y}\,dy = 2\int \frac{2}{2x-3}\,dx - \int \frac{1}{x-1}\,dx$$

so that now on the right hand side both integrals are of the fraction-pattern form. So, we can now integrate this:

$$\ln(y) = 2\ln(2x-3) - \ln(x-1) + \ln(C)$$

writing my constant as $\ln(C)$ rather than C. Why? Well, there there seemed to be a log-fest going on...

Using the log rules, we can combine all the terms on the right like this

$$\ln(y) = \ln\left(\frac{C(2x-3)^2}{x-1}\right)$$

so that

$$y = \frac{C(2x-3)^2}{x-1}$$

(c) Now if $y = 10$ when $x = 2$, then

$$10 = \frac{C(2\cdot 2-3)^2}{2-1}$$
$$10 = C$$

so our equation will be

$$y = \frac{10(2x-3)^2}{x-1}$$

Tips, Tricks and Patterns

Patterns:

$$\int f'(x)\cdot[f(x)]^n\,dx, \qquad \int f'(x)\cdot e^{f(x)}\,dx, \qquad \int \frac{f'(x)}{[f(x)]^n}\,dx \qquad \text{indeed...} \qquad \int f'(x)\cdot g[f(x)]\,dx$$

Question 8

The Question

$$I = \int_{x=0}^{x=5} e^{\sqrt{3x+1}}\,dx$$

(a) ...

(b) ...

(c) Use the substitution $t = \sqrt{3x+1}$ to show that I may be expressed as

$$I = \int_{t=a}^{t=b} kte^t\,dt$$

giving the values of a, b and k. **(5 marks)**

(d) Use integration by parts to evaluate this integral, and hence find the value of I correct to 4 significant figures, showing all the steps in your working. **(5 marks)**

My Answer

(c) Well, here they're asking us to use the substitution $t = \sqrt{3x+1}$. So it must be substitution! And we know what the substitution will be! Let's get on with it:

$$\text{If} \quad t = (3x+1)^{\frac{1}{2}} \quad \text{then} \quad \frac{dt}{dx} = \tfrac{1}{2}(3x+1)^{-\frac{1}{2}} \times 3 \quad \text{and} \quad dx = \tfrac{2}{3}(3x+1)^{\frac{1}{2}}\, dt = \tfrac{2}{3}t\, dt$$

using the chain rule to differentiate the substitution. Plugging this into the integral,

$$\begin{aligned} I &= \int_{x=0}^{x=5} e^t \cdot \tfrac{2}{3}t\, dt \\ &= \int_{x=0}^{x=5} \tfrac{2}{3}te^t\, dt \end{aligned}$$

Now for the limits. When $x = 0$, $t = \sqrt{3 \cdot 0 + 1} = 1$, and when $x = 5$, $t = \sqrt{3 \cdot 5 + 1} = 4$, so our integral becomes

$$I = \tfrac{2}{3} \int_{t=1}^{t=4} te^t\, dt$$

(d) Well, this is a classic integration by parts. Whenever we have a polynomial multiplied by a sine, cosine, exponential or log, it's parts. So...DIS is how we do it:

D	I	S
t	e^t	$+$
1	e^t	$-$
0	e^t	$+$
		$-$

Figure 9: Integrating $\int te^t\, dt$

So our integral becomes

$$\begin{aligned} I = \tfrac{2}{3} \int_{t=1}^{t=4} te^t\, dt &= \tfrac{2}{3}\left[te^t - e^t\right]_{t=1}^{t=4} \\ &= \tfrac{2}{3}\left[\left\{4e^4 - e^4\right\} - \left\{1e^1 - e^1\right\}\right] \\ &= \tfrac{2}{3} \cdot 3e^4 \\ &= 2e^4 \end{aligned}$$

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Question 2

The Question

Use the substitution $u = 2^x$ to find the exact value of

$$I = \int_{x=0}^{x=1} \frac{2^x}{(2^x+1)^2}\, dx$$

(6 marks)

My Answer

Well, this is probably a substitution! And I reckon the substitution is probably $u = 2^x$. Now the thing is, how do you differentiate $u = 2^x$?

You can't use the ploynomial rule to differentiate $u = 2^x$, as the polynomial rule (ax^n) requires the base (the x) to be the variable and the power (the n) to be the variable. But that's not what we've got with $u = 2^x$. So, what can you do?

In order to differentiate $u = 2^x$, we have to get the x out from being the power. The only way we know how to do that is by taking logs of both sides. Which log do we want? Well, the log that's easiest to differentiate is $\ln(x)$, so:

$$\begin{aligned} u &= 2^x \\ \ln(u) &= \ln(2^x) \\ \ln(u) &= x\ln(2) \end{aligned}$$

Now we differentiate implicitly:

$$\begin{aligned} \frac{1}{u}\frac{du}{dx} &= \ln(2) \\ \frac{du}{dx} &= u\ln(2) \\ \Big[\quad &= 2^x\ln(2) \Big] \end{aligned}$$

So that

$$dx = \frac{1}{u\ln(2)}\, du$$

Phew! That was tricky. Now we can shove the u stuff into our integral:

$$\begin{aligned} I &= \int_{x=0}^{x=1} \frac{u}{(u+1)^2} \cdot \frac{1}{u\ln(2)}\, du \\ &= \frac{1}{\ln(2)} \int_{x=0}^{x=1} \frac{1}{(u+1)^2}\, du \end{aligned}$$

Now how do we integrate this thing? Hang on a mo...the integrand looks a bit like $\frac{1}{x^2}$. So we need...another substitution!!!

If $\quad v = u+1 \quad$ then $\quad \frac{dv}{du} = 1 \quad$ and $\quad du = dv$

So

$$I = \frac{1}{\ln(2)} \int_{x=0}^{x=1} \frac{1}{v^2}\, dv$$

Ha!! Now we can integrate it:

$$\begin{aligned} I &= \frac{1}{\ln(2)}\Big[-v^{-1}\Big]_{x=0}^{x=1} \\ &= \frac{1}{\ln(2)}\Big[-\frac{1}{v}\Big]_{x=0}^{x=1} \\ &= \frac{1}{\ln(2)}\Big[-\frac{1}{u+1}\Big]_{x=0}^{x=1} \\ &= \frac{1}{\ln(2)}\Big[-\frac{1}{2^x+1}\Big]_{x=0}^{x=1} \\ &= \frac{1}{\ln(2)}\Big[\Big\{-\frac{1}{2^1+1}\Big\}-\Big\{-\frac{1}{2^0+1}\Big\}\Big] \\ &= \frac{1}{\ln(2)}\Big[\Big\{-\frac{1}{3}\Big\}-\Big\{-\frac{1}{2}\Big\}\Big] \\ &= \frac{1}{6\ln(2)} \end{aligned}$$

Tips, Tricks and Patterns

Tricks:

- Taking the $\ln(\cdot)$ of both sides and differentiating implicitly to differentiate $y = 2^x$.

Question 3

The Question

(a) Find

$$\int x\cos(2x)\ dx$$

(4 marks)

(b) Hence, using the identity $\cos(2x) = 2\cos^2(x) - 1$, deduce

$$\int x\cos^2(x)\ dx$$

(3 marks)

My Answer

(a) Well, this is a classic integration by parts. Whenever we have a polynomial multiplied by a sine, cosine, exponential or log, it's parts. So...see Figure 10 for DIS is how we do it:

D	I	S
x	$\cos(2x)$	$+$
1	$\frac{1}{2}\sin(2x)$	$-$
0	$-\frac{1}{4}\cos(2x)$	$+$
		$-$

Figure 10: Integrating $\int x\cos(2x)\ dx$

$$\begin{aligned} I &= \tfrac{1}{2}x\sin(2x) + \tfrac{1}{4}\cos(2x) + C \\ &= \tfrac{1}{2}\Big[x\sin(2x) + \tfrac{1}{2}\cos(2x)\Big] + C \end{aligned}$$

(b) Since $\cos(2x) = 2\cos^2(x) - 1$, then $\cos^2(x) = \frac{1}{2}[1 + \cos(2x)]$, and so

$$\begin{aligned} I = \int x\cos^2(x)\, dx &= \int x \cdot \tfrac{1}{2}[1 + \cos(2x)]\, dx \\ &= \tfrac{1}{2}\int x\,[1 + \cos(2x)]\, dx \\ &= \tfrac{1}{2}\int x + x\cos(2x)\, dx \\ &= \tfrac{1}{2}\int x\, dx + \tfrac{1}{2}\int x\cos(2x)\, dx \end{aligned}$$

So, using the result from part (a),

$$\begin{aligned} I &= \tfrac{1}{4}x^2 + \tfrac{1}{4}\left[x\sin(2x) + \tfrac{1}{2}\cos(2x)\right] + C \\ &= \tfrac{1}{4}\left[x^2 + x\sin(2x) + \tfrac{1}{2}\cos(2x)\right] + C \end{aligned}$$

Question 4

The Question

$$\frac{2(4x^2+1)}{(2x+1)(2x-1)} \equiv A + \frac{B}{2x+1} + \frac{C}{2x-1}$$

(a) Find the values of the constants A, B and C. **(4 marks)**

(b) Hence show that the exact value of

$$I = \int_{x=1}^{x=2} \frac{2(4x^2+1)}{(2x+1)(2x-1)}\, dx$$

is $2 + \ln(k)$, giving the value of the constant k. **(6 marks)**

My Answer

(a) This turns out to be

$$\frac{2(4x^2+1)}{(2x+1)(2x-1)} \equiv 2 - \frac{2}{2x+1} + \frac{2}{2x-1}$$

(b) So,

$$\begin{aligned} I &= \int_{x=1}^{x=2} 2 - \frac{2}{2x+1} + \frac{2}{2x-1}\, dx \\ &= \int_{x=1}^{x=2} 2\, dx - \int_{x=1}^{x=2} \frac{2}{2x+1}\, dx + \int_{x=1}^{x=2} \frac{2}{2x-1}\, dx \end{aligned}$$

Now **all** partial fractions integrals end up with terms like this

$$\int \frac{a}{bx+c}\, dx$$

to integrate. And there is a standard way that you do these. When you learn about using substitution to solve integrals, there is a very common family of integrals that you learn about. These integrals are

$$\int f'(x) \cdot [f(x)]^n\, dx, \qquad \int f'(x) \cdot e^{f(x)}\, dx, \qquad \int \frac{f'(x)}{[f(x)]^n}\, dx, \qquad \text{indeed...} \qquad \int f'(x) \cdot g[f(x)]\, dx$$

where the coloured bits are the things to look out for. If you have an integral of one of these types, then you use substitution to transform the integral into a standard form, and the substitution is $u = f(x)$.

And it turns out that (see Section 1, Question 3)

$$\int \frac{f'(x)}{f(x)}\, dx = \ln[f(x)]$$

So, noticing that in the integrals with fractions, the top is the differential of the bottom, then

$$
\begin{aligned}
I &= \Big[2x - \ln(2x+1) + \ln(2x-1)\Big]_{x=1}^{x=2} \\
&= \Big[\Big\{2x - \ln(2x+1) + \ln(2x-1)\Big\} - \Big\{2x - \ln(2x+1) + \ln(2x-1)\Big\}\Big] \\
&= \Big[\Big\{2\, cdot 2 - \ln(2\cdot 2+1) + \ln(2\cdot 2-1)\Big\} - \Big\{2\cdot 1 - \ln(2\cdot 1+1) + \ln(2\cdot 1-1)\Big\}\Big] \\
&= \Big[\Big\{4 - \ln(5) + \ln(3)\Big\} - \Big\{2 - \ln(3) + \ln(1)\Big\}\Big] \\
&= 2 - \ln(5) + 2\ln(3) \\
&= 2 - \ln(5) + \ln(3^2) \\
&= 2 + \ln\left(\tfrac{9}{5}\right)
\end{aligned}
$$

Tips, Tricks and Patterns

Patterns:

$$\int f'(x)\cdot[f(x)]^n\,dx, \qquad \int f'(x)\cdot e^{f(x)}\,dx, \qquad \int \frac{f'(x)}{[f(x)]^n}\,dx \qquad \text{indeed...} \qquad \int f'(x)\cdot g[f(x)]\,dx$$

Question 7

The Question

Figure 11 shows part of the curve with the equation $y = \sqrt{\tan(x)}$. The finite region R, which is bounded by the curve, the x-axis and the line $x = \frac{\pi}{4}$, is shown shaded in Figure 11.

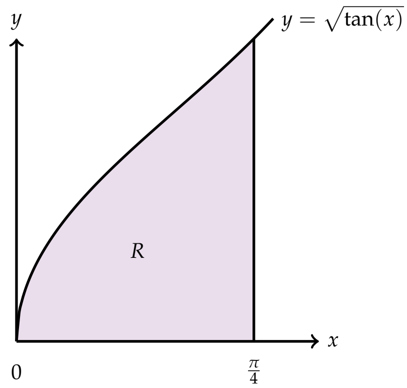

Figure 11: A graph of $y = \sqrt{\tan(x)}$

(a) ...

(b) ...

The region R is rotated through 2π radians around the x-axis to generate a solid of revolution.

(c) Use integration to find an exact value for the volume of the solid generated. **(4 marks)**

My Answer

(c) The integral we want will be

$$
\begin{aligned}
V &= \pi \int_{x=0}^{x=\frac{\pi}{4}} y^2\,dx \\
&= \pi \int_{x=0}^{x=\frac{\pi}{4}} \tan(x)\,dx
\end{aligned}
$$

Hang on...the integral of $\tan(x)$ is in the formula book!

$$V = \pi \int_{x=0}^{x=\frac{\pi}{4}} y^2 \, dx$$
$$= \pi \Big[\ln\left(|\sec(x)|\right) \Big]_{x=0}^{x=\frac{\pi}{4}}$$

Now we can plug the numbers in:

$$V = \pi \left[\left\{ \ln\left(\left| \sec\left(\frac{\pi}{4}\right) \right| \right) \right\} - \left\{ \ln\left(|\sec(0)|\right) \right\} \right]$$
$$= \pi \left[\left\{ \ln\left(\sqrt{2}\right) \right\} - \left\{ \ln(1) \right\} \right]$$
$$= \pi \ln\left(\sqrt{2}\right)$$
$$= \pi \ln\left(2^{\frac{1}{2}}\right)$$
$$= \frac{\pi}{2} \ln(2)$$

Question 8

The Question

A population growth is modelled by the differential equation

$$\frac{dP}{dt} = kP$$

where P is the population, t is the time measured in days and k is a positive constant.

Given that the initial population is P_0,

(a) solve the differential equation, giving P in terms of P_0, k and t. **(4 marks)**

(b) ...

In an improved model the differential equation is given as

$$\frac{dP}{dt} = \lambda P \cos(\lambda t)$$

where P is the population, t is the time measured in days and λ is a positive constant.

Given, again, that the initial population is P_0 and that time is measured in days,

(c) solve the second differential equation, giving P in terms of P_0, λ and t. **(4 marks)**

Given also that $\lambda = 2.5$,

(d) find the time taken, to the nearest minute, for the population to reach $2P_0$ for the first time, using the improved model. **(3 marks)**

My Answer

(a) The only way we know how to solve differential equations is to separate the variables,

$$\frac{1}{P} \, dP = k \, dt$$

and integrate both sides

$$\int \frac{1}{P} \, dP = \int k \, dt$$

Both sides of this equation are standard forms, so we can integrate them:

$$\ln(P) = kt + \ln(C)$$

writing $\ln(C)$ instead of C. Why? Experience! When $t = 0$, $P = P_0$, so

$$\ln(P_0) = \ln(C)$$

and so $C = P_0$. Simples! So our equation is

$$\begin{aligned}
\ln(P) &= kt + \ln(P_0) \\
\ln(P) - \ln(P_0) &= kt \\
\ln\left(\frac{P}{P_0}\right) &= kt \\
\frac{P}{P_0} &= e^{kt} \\
P &= P_0 e^{kt}
\end{aligned}$$

(b) Again, separate the variables

$$\frac{1}{P}\ dP = \lambda \cos(\lambda t)\ dt$$

and integrate both sides

$$\int \frac{1}{P}\ dP = \int \lambda \cos(\lambda t)\ dt$$

Again, both integrals here are standard forms. Well, almost! The one on the right is technically a substitution, but I'm hoping you can do it without the formal substitution malarkey:

$$\begin{aligned}
\ln(P) &= \lambda \cdot \frac{1}{\lambda} \sin(\lambda t) + C \\
&= \sin(\lambda t) + C
\end{aligned}$$

Again, when $t = 0$, $P = P_0$, so

$$\begin{aligned}
\ln(P_0) &= \sin(\lambda \cdot 0) + C \\
\ln(P_0) &= C
\end{aligned}$$

so our equation becomes

$$\begin{aligned}
\ln(P) &= \sin(\lambda t) + \ln(P_0) \\
\ln(P) - \ln(P_0) &= \sin(\lambda t) \\
\ln\left(\frac{P}{P_0}\right) &= \sin(\lambda t) \\
\frac{P}{P_0} &= e^{\sin(\lambda t)} \\
P &= P_0 e^{\sin(\lambda t)}
\end{aligned}$$

(d) If $\lambda = 2.5$, then our improved model is

$$P = P_0 e^{\sin(2.5t)}$$

So to find the time when $P = 2P_0$, we have to solve the equation

$$2P_0 = P_0 e^{\sin(2.5t)}$$

and so

$$\begin{aligned}
2 &= e^{\sin(2.5t)} \\
\ln(2) &= \sin(2.5t)
\end{aligned}$$

So

$$\begin{aligned}
2.5t &= \sin^{-1}(\ln[2]) \\
t &= \frac{1}{2.5} \sin^{-1}(\ln[2]) \\
t &= 0.306338...\text{days} \\
t &= 0.306338... \times 24 = 7.35...\text{hours} \\
t &= 7.35... \times 60 = 441\text{minutes, to the nearest minute}
\end{aligned}$$

[Technically, I should draw thae graph of the sine function to clearly show that this will be the *first* solution...]

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Question 3

The Question

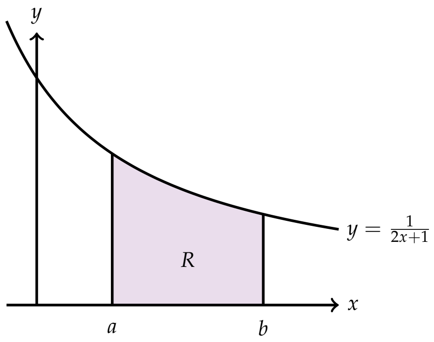

Figure 12: A graph of $y = \frac{1}{2x+1}$

The curve shown in Figure 12 has the equation $y = \frac{1}{2x+1}$. The finite region R bounded by the curve, the x-axis and the lines $x = a$ and $x = b$ is shown in Figure 12. This region is rotated through 360° about the x-axis to generate a solid of revolution.

Find the volume of the solid generated. Express your answer as a single simplified fraction, in terms of a and b. **(5 marks)**

My Answer

The integral we want will be

$$V = \pi \int_{x=a}^{x=b} \frac{1}{(2x+1)^2}\, dx$$

Now the integrand reminds me of $\frac{1}{x^2}$. Aha! Let's try this substitution then:

$$\text{If} \qquad u = 2x+1 \qquad \text{then} \qquad \frac{du}{dx} = 2 \qquad \text{and} \qquad dx = \frac{1}{2}\, du$$

Plugging this into our integral

$$\begin{aligned} V &= \pi \int_{x=a}^{x=b} \frac{1}{u^2} \cdot \frac{1}{2}\, du \\ &= \frac{\pi}{2} \int_{x=a}^{x=b} u^{-2}\, du \end{aligned}$$

This is now a standard form, so we can integrate it:

$$\begin{aligned} V &= \frac{\pi}{2}\Big[-\frac{1}{u}\Big]_{x=a}^{x=b} \\ &= \frac{\pi}{2}\Big[-\frac{1}{2x+1}\Big]_{x=a}^{x=b} \\ &= \frac{\pi}{2}\Big[\Big\{-\frac{1}{2b+1}\Big\} - \Big\{-\frac{1}{2a+1}\Big\}\Big] \end{aligned}$$

From here on, it's a bit of an algebra exercise...

$$\begin{aligned} V &= \frac{\pi}{2}\left[\frac{1}{2a+1} - \frac{1}{2b+1}\right] \\ &= \frac{\pi}{2}\left[\frac{2b+1}{(2a+1)(2b+1)} - \frac{2a+1}{(2a+1)(2b+1)}\right] \\ &= \frac{\pi}{2}\left[\frac{2b-2a}{(2a+1)(2b+1)}\right] \\ &= \frac{\pi(b-a)}{(2a+1)(2b+1)} \end{aligned}$$

Tips, Tricks and Patterns

Tips:

- What standard form is this integral closest to?

Question 4

The Question

(i) Find

$$I = \int \ln\left(\tfrac{1}{2}x\right)\, dx$$

(4 marks)

(ii) Find the exact value of

$$I = \int_{x=\frac{\pi}{4}}^{x=\frac{\pi}{2}} \sin^2(x)\, dx$$

(5 marks)

My Answer

(a) Well, this is a classic integration by parts. Whenever we have a polynomial multiplied by a sine, cosine, exponential or log, it's parts. And even when you just have a log to integrate, that's parts too! But in this integral, there's no product, so how can we use parts? Ah...we can make one by a cunning trick: you multiply the integrand by 1!! And because we don't know how to integrate log, we have to put that in the D column. So...check out Figure 13 for DIS is how we do it:

D	I	S
$\ln\left(\frac{1}{2}x\right)$	1	+
$\frac{1}{x}$	x	−
		+

Figure 13: Integrating $\int \ln\left(\frac{1}{2}x\right)\, dx$

So our integral becomes

$$\begin{aligned} I &= x\ln\left(\tfrac{1}{2}x\right) - \int \frac{1}{x}\cdot x\, dx \\ &= x\ln\left(\tfrac{1}{2}x\right) - \int 1\, dx \\ &= x\ln\left(\tfrac{1}{2}x\right) - x + C \end{aligned}$$

(b) When you have $\sin^2(x)$ or $\cos^2(x)$ to integrate, there is a trick. It's not easy to integrate powers of $\sin(x)$ and $\cos(x)$ higher than one, so we have to convert them to single powers using trigonometrical identities.

And the identities are: $\cos(2x) \equiv 2\cos^2(x) - 1 \equiv 1 - 2\sin^2(x)$ (from the addition formulae). Using these identities we can convert either $\sin^2(x)$ or $\cos^2(x)$ into something with a $\cos(2x)$ in it.

How does this help here? Well, let's see what happens when we apply this trigonometrical identity trick to our integral. Since $\cos(2x) \equiv 1 - 2\sin^2(x)$, then $\sin^2(x) \equiv \frac{1}{2}\left[1 - \cos(2x)\right]$, so

$$\begin{aligned} I &= \int_{x=\frac{\pi}{4}}^{x=\frac{\pi}{2}} \tfrac{1}{2}\left[1 - \cos(2x)\right]\, dx \\ &= \tfrac{1}{2}\int_{x=\frac{\pi}{4}}^{x=\frac{\pi}{2}} 1 - \cos(2x)\, dx \end{aligned}$$

Plugging the numbers in:

$$\begin{aligned}
I = &= \tfrac{1}{2}\left[x - \tfrac{1}{2}\sin(2x)\right]_{x=\frac{\pi}{4}}^{x=\frac{\pi}{2}} \\
&= \tfrac{1}{2}\left[\left\{\tfrac{\pi}{2} - \tfrac{1}{2}\sin\left(2\cdot\tfrac{\pi}{2}\right)\right\} - \left\{\tfrac{\pi}{4} - \tfrac{1}{2}\sin\left(2\cdot\tfrac{\pi}{4}\right)\right\}\right] \\
&= \tfrac{1}{2}\left[\left\{\tfrac{\pi}{2} - 0\right\} - \left\{\tfrac{\pi}{4} - \tfrac{1}{2}\cdot 1\right\}\right] \\
&= \tfrac{1}{2}\left[\tfrac{\pi}{2} - \tfrac{\pi}{4} + \tfrac{1}{2}\right] \\
&= \tfrac{1}{2}\left[\tfrac{\pi}{2} - \tfrac{\pi}{4} + \tfrac{1}{2}\right] \\
&= \tfrac{1}{2}\left[\tfrac{\pi}{4} + \tfrac{1}{2}\right] \\
&= \tfrac{1}{4}\left[\tfrac{\pi}{2} + 1\right]
\end{aligned}$$

Tips, Tricks and Patterns

Tricks:

- Multiplying by 1 to make a product!
- When you have $\sin^2(x)$ or $\cos^2(x)$ to integrate, there is a trick.
  You have to use the identities: $\mathbf{\cos(2x) \equiv 2\cos^2(x) - 1 \equiv 1 - 2\sin^2(x)}$.

Question 7

The Question

The curve C has parametric equations

$$x = \ln(t+2), \qquad y = \frac{1}{t+1}, \qquad t > -1$$

The finite region R between the curve C and the x-axis, bounded by the lines with equations $x = \ln(2)$ and $x = \ln(4)$, is shown shaded in Figure 14.

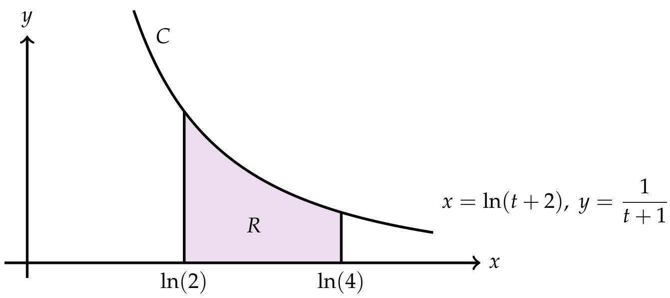

Figure 14: A graph of $x = \ln(t+2),\ y = \dfrac{1}{t+1}$

(a) Show that the area of R is given by the integral

$$\int_{t=0}^{t=2} \frac{1}{(t+1)(t+2)}\, dt$$

(4 marks)

(b) Hence find an exact value for this area. **(6 marks)**

(c) ...

(d) ...

My Answer

(a) The integral we want will be

$$I = \int_{x=\ln(2)}^{x=\ln(4)} y \, dx$$

Now we need to express everything in the integral in terms of t. Well, the y is no problem: that's just going to be $\frac{1}{t+1}$. But what about the dx? We will need to transform that into dt. How? Well, how can we find a connection between dx and dt? Differentiate the $x = \ln(t+2)$ formula, of course!

$$x = \ln(t+2)$$
$$\frac{dx}{dt} = \frac{1}{t+2} \cdot 1$$

using the chain rule, so

$$dx = \frac{1}{t+2} \, dt$$

Now we just have the limits to figure out. Now when $x = \ln(2)$, t must be 0 (just looking at the formula for x), and when $x = \ln(4)$, t must be 2. So, our integral becomes

$$I = \int_{t=0}^{t=2} \frac{1}{t+1} \cdot \frac{1}{t+2} \, dt$$

as required!!

(b) Well, this is a classic partial fractions problem. How can you tell? We have a fraction with factors in the denominator. So we need to find the A and B that makes this work:

$$\frac{1}{(t+1)(t+2)} = \frac{A}{t+1} + \frac{B}{t+2}$$

One way to do that is to add the fractions on the right hand side, and compare what you get with the left hand side:

$$\frac{1}{(t+1)(t+2)} = \frac{A(t+2)}{(t+1)(t+2)} + \frac{B(t+1)}{(t+1)(t+2)}$$
$$= \frac{A(t+2) + B(t+1)}{(t+1)(t+2)}$$

Now if we compare the numerators,

$$1 = A(t+2) + B(t+1)$$

Now this must be true for all values of t, so we can pick a value that's convenient. Let's say $t = -1$. In that case, we get

$$1 = A(-1+2) + B(-1+1)$$

in which case $A = 1$. And now let's say $t = -2$. In that case, we get

$$1 = A(-2+2) + B(-2+1)$$

in which case $B = -1$.

So we have found that

$$\frac{1}{(t+1)(t+2)} = \frac{1}{t+1} - \frac{1}{t+2}$$

so we can transform our integral to

$$I = \int_{t=0}^{t=2} \frac{1}{t+1} - \frac{1}{t+2} \, dt$$

or indeed

$$I = \int_{t=0}^{t=2} \frac{1}{t+1} \, dt - \int_{t=0}^{t=2} \frac{1}{t+2} \, dt$$

Now **all** partial fractions integrals end up with terms like this

$$\int \frac{a}{bx+c} \, dx$$

to integrate. And there is a standard way that you do these. When you learn about using substitution to solve integrals, there is a very common family of integrals that you learn about. These integrals are

$$\int f'(x) \cdot [f(x)]^n \, dx, \qquad \int f'(x) \cdot e^{f(x)} \, dx, \qquad \int \frac{f'(x)}{[f(x)]^n} \, dx, \qquad \text{indeed...} \qquad \int f'(x) \cdot g[f(x)] \, dx$$

where the coloured bits are the things to look out for. If you have an integral of one of these types, then you use substitution to transform the integral into a standard form, and the substitution is $u = f(x)$.

The upshot of all this is that if you have an integral that looks like the left hand side of this

$$\int \frac{f'(x)}{f(x)} \, dx = \ln [f(x)]$$

then the answer is just $\ln [f(x)]$.

Now in our integrals, the tops of both integrands is the differential of the bottom! so we can just write the answer down:

$$\begin{aligned} I &= [\ln(t+1)]_{t=0}^{t=2} - [\ln(t+2)]_{t=0}^{t=2} \\ &= [\ln(t+1) - \ln(t+2)]_{t=0}^{t=2} \\ &= \left[\ln\left(\frac{t+1}{t+2}\right)\right]_{t=0}^{t=2} \end{aligned}$$

And now we can plug the numbers in

$$\begin{aligned} I &= \ln\left(\frac{2+1}{2+2}\right) - \ln\left(\frac{0+1}{0+2}\right) \\ &= \ln\left(\frac{3}{4}\right) - \ln\left(\frac{1}{2}\right) \\ &= \ln\left(\frac{\frac{3}{4}}{\frac{1}{2}}\right) \\ &= \ln\left(\frac{3}{2}\right) \end{aligned}$$

Tips, Tricks and Patterns

Patterns:

$$\int f'(x) \cdot [f(x)]^n \, dx, \qquad \int f'(x) \cdot e^{f(x)} \, dx, \qquad \int \frac{f'(x)}{[f(x)]^n} \, dx \qquad \text{indeed...} \qquad \int f'(x) \cdot g[f(x)] \, dx$$

Question 8

The Question

Liquid is pouring into a large vertical circular cylinder at a constant rate of 1600 cm^3 s^{-1} and is leaking out of a hole in the base, at a rate proportional to the square root of the height of the liquid already in the cylinder. The area of the circular cross section of the cylinder is 4000 cm^2.

(a) Show that at time t seconds, the height h cm of liquid in the cylinder satisfies the differential equation

$$\frac{dh}{dt} = 0.4 - k\sqrt{h}$$

where k is a positive constant. **(3 marks)**

When $h = 25$, water is leaking out of the hole at a rate of 400 cm^3 s^{-1}.

(b) Show that $k = 0.02$. **(1 mark)**

(c) Separate the variables the variables of the differential equation

$$\frac{dh}{dt} = 0.4 - 0.02\sqrt{h}$$

to show that the time taken to fill the cylinder from empty to a height of 100 cm is given by

$$\int_{h=0}^{h=100} \frac{50}{20 - \sqrt{h}} \, dh$$

(2 marks)

Using the substitution $h = (20 - x)^2$, or otherwise,

(d) find the exact value of

$$\int_{h=0}^{h=100} \frac{50}{20 - \sqrt{h}}\, dh$$

(6 marks)

(e) Hence find the time taken to fill the cylinder from empty to a height of 100 cm, giving your answer in minutes and seconds to the nearest second. **(1 mark)**

My Answer

(c) The only way we can solve differential equations is to separate the variables, and integrate both sides. But first, I want to simplify the numbers in this differential equation a bit. Looking at the answer, and noticing that $0.02 = \frac{1}{50}$, the first thing I want to do here is to multiply both sides by 50:

$$50\frac{dh}{dt} = 20 - \sqrt{h}$$

That's a bit better. Now...separate the variables:

$$\frac{50}{20 - \sqrt{h}}\, dh = dt$$

and integrate both sides

$$\int \frac{50}{20 - \sqrt{h}}\, dh = \int\, dt$$

Now, what about the limits? Well, clearly,

$$\int_{h=0}^{h=100} \frac{50}{20 - \sqrt{h}}\, dh = T$$

where T will be the time it takes for the height to get from 0 to 100 cm.

(d) Well, this will be a substitution (because they're telling us it will be), and the substitution will be $h = (20 - x)^2$. That means that $\sqrt{h} = 20 - x$ and

If $\quad h = (20 - x)^2 \quad$ then $\quad \frac{dh}{dx} = 2(20 - x) \cdot -1 \quad$ and $\quad dh = -2(20 - x)\, dx$

using the chain rule. So, plugging this into our integral,

$$\begin{aligned} T &= \int_{h=0}^{h=100} \frac{50}{20 - (20 - x)} \cdot -2(20 - x)\, dx \\ &= -2 \int_{h=0}^{h=100} \frac{50(20 - x)}{x}\, dx \end{aligned}$$

There's a standard bit of algebra we do in this kind of situation, when we have a single variable in the denominator. Just multiply everything out and use the addition rule for fractions backwards:

$$\begin{aligned} T &= -2 \int_{h=0}^{h=100} \frac{1000 - 50x}{x}\, dx \\ &= -2 \int_{h=0}^{h=100} \frac{1000}{x} - \frac{50x}{x}\, dx \\ &= -2 \int_{h=0}^{h=100} \frac{1000}{x} - 50\, dx \end{aligned}$$

Now we have nothing but standard forms in the integral. So we can integrate it:

$$\begin{aligned}
T &= -2\Big[1000\ln(x) - 50x\Big]_{h=0}^{h=100} \\
&= -2\Big[1000\ln(20-\sqrt{h}) - 50(20-\sqrt{h})\Big]_{h=0}^{h=100} \\
&= -2\Big[\Big\{1000\ln(20-\sqrt{100}) - 50(20-\sqrt{100})\Big\} - \Big\{1000\ln(20-\sqrt{0}) - 50(20-\sqrt{0})\Big\}\Big] \\
&= -2\Big[\Big\{1000\ln(10) - 500\Big\} - \Big\{1000\ln(20) - 1000\Big\}\Big] \\
&= -2\Big[1000\ln\left(\tfrac{1}{2}\right) + 500\Big] \\
&= -2000\ln\left(\tfrac{1}{2}\right) - 1000 \\
&= 2000\ln(2) - 1000 \\
&\approx 386 \text{ seconds (to the nearest second)}
\end{aligned}$$

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Question 2

The Question

(a) Use integration by parts to find

$$I = \int xe^x \, dx$$

(3 marks)

(b) Hence find

$$I = \int x^2 e^x \, dx$$

(3 marks)

My Answer

(a) Well, this is a classic integration by parts. Whenever we have a polynomial multiplied by a sine, cosine, exponential or log, it's parts. So...DIS is how we do it:

D	I	S
x	e^x	$+$
1	e^x	$-$
0	e^x	$+$
		$-$

Figure 15: Integrating $\int xe^x \, dx$

So our integral becomes

$$I = xe^x - e^x + C$$

(b) And...check out Figure 16 for DIS is how we do it again:

D	I	S
x^2	e^x	$+$
$2x$	e^x	$-$
2	e^x	$+$
0	e^x	$-$
		$+$

Figure 16: Integrating $\int x^2 e^x \, dx$

So our integral becomes

$$I = x^2 e^x - 2xe^x + 2e^x + C$$

Question 7

The Question

(a) Express

$$\frac{2}{4-y^2}$$

in partial fractions. **(3 marks)**

(b) Hence obtain the solution of

$$2\cot(x)\frac{dy}{dx} = 4 - y^2$$

for which $y = 0$ at $x = \frac{\pi}{3}$, giving your answer in the form $\sec^2(x) = g(y)$. **(8 marks)**

My Answer

(a) Noticing that the denominator of this fraction is a *difference of two squares*, we can write it as

$$\frac{2}{4-y^2} = \frac{A}{2+y} + \frac{B}{2-y}$$

Adding the two fractions together on the right we get

$$\begin{aligned}\frac{2}{4-y^2} &= \frac{A(2-y)}{(2+y)(2-y)} + \frac{B(2+y)}{(2+y)(2-y)} \\ &= \frac{A(2-y)+B(2+y)}{(2+y)(2-y)}\end{aligned}$$

And then comparing the numerators we get

$$2 = A(2-y) + B(2+y)$$

Now this has to be true for all values of y, so we can pick convenient ones to serve our purpose. For example, if we choose $y = 2$ then

$$2 = A(2-2) + B(2+2)$$

in which case $B = \frac{1}{2}$. And if we choose $y = -2$ then

$$2 = A(2+2) + B(2+-2)$$

in which case $A = \frac{1}{2}$. So:

$$\frac{2}{4-y^2} = \frac{1}{2}\left[\frac{1}{2+y} + \frac{1}{2-y}\right]$$

(b) The only way we can solve differential equations is to separate the variables

$$\frac{2}{4-y^2}\ dy = \tan(x)\ dx$$

and integrate both sides

$$\int \frac{2}{4-y^2}\ dy = \int \tan(x)\ dx$$

Now from part (a) we can write the left hand side like this

$$\frac{1}{2}\int \frac{1}{2+y} + \frac{1}{2-y}\ dy = \int \tan(x)\ dx$$

or indeed

$$\frac{1}{2}\int \frac{1}{2+y}\ dy - \frac{1}{2}\int \frac{-1}{2-y}\ dy = \int \tan(x)\ dx$$

splitting the inegral on the left hand side into two bits. Now why on earth have I written the $\frac{1}{2-y}$ integral as $-\frac{-1}{2-y}$?? Well, that's because on the left hand side we have two fractions, and in each case the top is now the differential of the bottom! That rings a bell! If it doesn't, go back to a previous integration using partial fractions for a fuller explanation. We should now be able to just write down the integrations:

$$\tfrac{1}{2}\ln(2+y) - \tfrac{1}{2}\ln(2-y) = \ln(|\sec(x)|) + C$$

Now to find the C we can put the boundary condition in: when $x = \frac{\pi}{3}$, $y = 0$:

$$\begin{aligned}
\tfrac{1}{2}\ln(2+0) - \tfrac{1}{2}\ln(2-0) &= \ln\left(\left|\sec\left(\tfrac{\pi}{3}\right)\right|\right) + C \\
\tfrac{1}{2}\ln(2) - \tfrac{1}{2}\ln(2) &= \ln(2) + C \\
0 &= \ln(2) + C \\
C &= -\ln(2)
\end{aligned}$$

So,

$$\begin{aligned}
\tfrac{1}{2}\ln(2+y) - \tfrac{1}{2}\ln(2-y) &= \ln(|\sec(x)|) - \ln(2) \\
\tfrac{1}{2}\left[\ln(2+y) - \ln(2-y)\right] + \ln(2) &= \ln(|\sec(x)|) \\
\frac{1}{2}\left[\ln\left(\frac{2+y}{2-y}\right) + 2\ln(2)\right] &= \ln(|\sec(x)|) \\
\frac{1}{2}\left[\ln\left(\frac{2+y}{2-y}\right) + \ln\left(2^2\right)\right] &= \ln(|\sec(x)|) \\
\frac{1}{2}\left[\ln\left(\frac{4(2+y)}{2-y}\right)\right] &= \ln(|\sec(x)|) \\
\ln\left(\sqrt{\frac{4(2+y)}{2-y}}\right) &= \ln(|\sec(x)|)
\end{aligned}$$

and so

$$\begin{aligned}
\sqrt{\frac{4(2+y)}{2-y}} &= \sec(x) \\
\frac{4(2+y)}{2-y} &= \sec^2(x)
\end{aligned}$$

Question 8

The Question

Figure 17 shows the curve C with parametric equations

$$x = 8\cos(t), \qquad y = 4\sin(2t), \qquad 0 \le t \le \tfrac{\pi}{2}$$

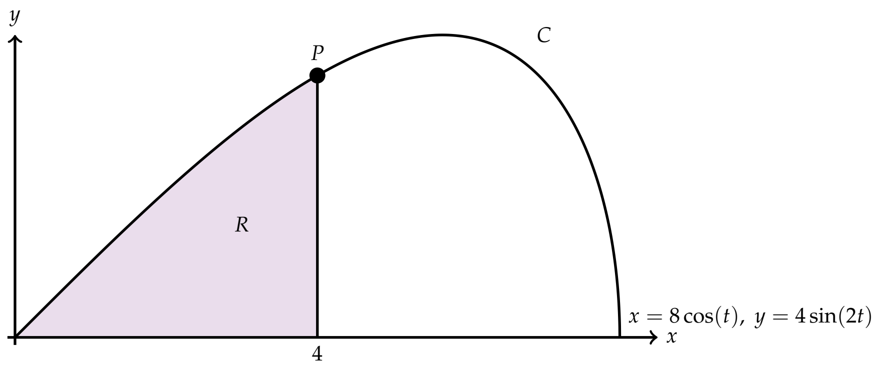

Figure 17: A graph of $x = 8\cos(t),\ y = 4\sin(2t)$

The point P lies on C and has coordinates $(4, 2\sqrt{3})$.

(a) Find the value of t at the point P. **(2 marks)**

The line l is a normal to C at P.

(b) ...

The finite region R enclosed by the curve C, the x-axis and the line $x = 4$ is shown shaded in Figure 17.

(c) Show that the area of R is given by the integral

$$\int_{x=\frac{\pi}{3}}^{x=\frac{\pi}{2}} 64\sin^2(t)\cos(t)\,dt$$

(4 marks)

(d) Use this integral to find the area of R, giving your answer in the form $a + b\sqrt{3}$, where a and b are constants to be determined. **(4 marks)**

My Answer

(a) If $x = 4$ at P, then $4 = 8\cos(t)$. So $\cos(t) = \frac{1}{2}$, and so $t = \frac{\pi}{6}$ at P, as this is the only value of t that would give $\cos(t) = \frac{1}{2}$ in the given range of values for t.

(b) ...

(c) The integral we want is

$$I = \int_{x=0}^{x=4} y\ dx$$

And we have to convert this to t stuff. The y is easy: that's just $y = 4\sin(2t)$. To find dx we have to differentiate the x equation. When we do that we get $\frac{dx}{dt} = -8\sin(t)$, so that $dx = -8\sin(t)\ dt$. Now for the limits. We've already discovered that when $x = 4$, $t = \frac{\pi}{3}$. And when $x = 0$, $8\cos(t) = 0$, which means that $t = \frac{\pi}{2}$. Again, this is the only value of t that would give $\cos(t) = 0$ in the given range of values for t. So, our integral becomes

$$I = -\int_{t=\frac{\pi}{2}}^{t=\frac{\pi}{3}} 4\sin(2t) \cdot -8\sin(t)\ dt$$

Now in the answer, there is no $2t$ angle, so we need to use the double angle identity $\sin(2x) \equiv 2\sin(x)\cos(x)$ to help us here. Plugging that in we get

$$\begin{aligned} I &= -\int_{t=\frac{\pi}{2}}^{t=\frac{\pi}{3}} 4 \cdot 2\sin(t)\cos(t) \cdot -8\sin(t)\ dt \\ &= -64\int_{t=\frac{\pi}{2}}^{t=\frac{\pi}{3}} \sin^2(t)\cos(t)\ dt \end{aligned}$$

Now the only thing left here is to sort out the $-$ sign and the limits. Now if you think about it a bit, and $\int f(x)\ dx = F(x) + C$, then

$$\int_{x=a}^{x=b} f(x)\ dx = F(b) - F(a) = -\Big[F(a) - F(b)\Big] = -\int_{x=b}^{x=a} f(x)\ dx$$

In other words, if you swap limits on an integral, you need to stick a $-$ sign out the front. So we can write our integral as Plugging that in we get

$$I = 64\int_{t=\frac{\pi}{3}}^{t=\frac{\pi}{2}} \sin^2(t)\cos(t)\ dt$$

as required.

(d) Now I've noticed here that if you differentiate $\sin(t)$ you get $\cos(t)$. When you learn about using substitution to solve integrals, there is a very common family of integrals that you learn about. These integrals are

$$\int f'(x) \cdot [f(x)]^n\ dx, \qquad \int f'(x) \cdot e^{f(x)}\ dx, \qquad \int \frac{f'(x)}{[f(x)]^n}\ dx, \qquad \text{indeed...} \qquad \int f'(x) \cdot g[f(x)]\ dx$$

where the coloured bits are the things to look out for. If you have an integral of one of these types, then you use substitution to transform the integral into a standard form, and the substitution is $u = f(x)$.

So we are going to use substitution to solve this integral, and the substitution is $u = \sin(t)$:

If $\quad u = \sin(t) \quad$ then $\quad \frac{du}{dt} = \cos(t) \quad$ and $\quad dt = \frac{1}{\cos(t)}\ dx$

Plugging this stuff into our integral gives

$$\begin{aligned} I &= 64 \int_{t=\frac{\pi}{3}}^{t=\frac{\pi}{2}} u^2 \cos(t) \cdot \frac{1}{\cos(t)}\, dx \\ &= 64 \int_{t=\frac{\pi}{3}}^{t=\frac{\pi}{2}} u^2\, dx \end{aligned}$$

This integrand is now a standard form, so we can integrate it

$$\begin{aligned} I &= 64 \left[\tfrac{1}{3}u^3\right]_{t=\frac{\pi}{3}}^{t=\frac{\pi}{2}} \\ &= 64 \left[\tfrac{1}{3}\sin^3(t)\right]_{t=\frac{\pi}{3}}^{t=\frac{\pi}{2}} \\ &= 64 \left[\left\{\tfrac{1}{3}\sin^3\left(\tfrac{\pi}{2}\right)\right\} - \left\{\tfrac{1}{3}\sin^3\left(\tfrac{\pi}{3}\right)\right\}\right] \\ &= 64 \left[\left\{\tfrac{1}{3}\cdot 1\right\} - \left\{\tfrac{1}{3}\tfrac{3\sqrt{3}}{8}\right\}\right] \\ &= 64 \left[\tfrac{1}{3} - \tfrac{\sqrt{3}}{8}\right] \\ &= \tfrac{64}{3} - 8\sqrt{3} \end{aligned}$$

Tips, Tricks and Patterns

Tricks:

- If you swap limits on an integral, you need to stick a $-$ sign out the front.

Patterns:

$$\int f'(x) \cdot [f(x)]^n\, dx, \qquad \int f'(x) \cdot e^{f(x)}\, dx, \qquad \int \frac{f'(x)}{[f(x)]^n}\, dx \qquad \text{indeed...} \qquad \int f'(x) \cdot g[f(x)]\, dx$$

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8 Edexcel C4 January 2009

Question 2

The Question

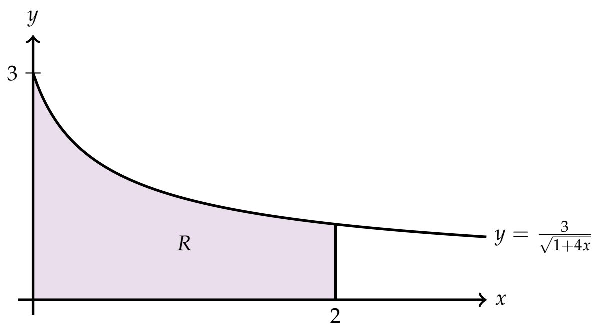

Figure 18: A graph of $y = \frac{3}{\sqrt{1+4x}}$

Figure 18 shows part of the curve $y = \frac{3}{\sqrt{1+4x}}$. The region R is bounded by the curve, the x-axis, and the lines $x = 0$ and $x = 2$, as shown in Figure 18.

(a) Use integration to find the area of R. **(4 marks)**

The region R is rotated $360°$ about the x-axis.

(b) Use integration to find the exact value of the solid formed. **(5 marks)**

My Answer

(a) The integral we want will be

$$I = \int_{x=0}^{x=2} \frac{3}{\sqrt{1+4x}}\, dx$$

To me, this looks a bit like $x^{-\frac{1}{2}}$. So, let's try the substitution $u = 1 + 4x$.

If $u = 1 + 4x$ then $\frac{du}{dx} = 4$ and $dx = \frac{1}{4}\, du$

Plugging this into our integral gives

$$I = \int_{x=0}^{x=2} \frac{3}{u^{\frac{1}{2}}} \cdot \frac{1}{4}\, du$$

or in other words

$$I = \frac{3}{4} \int_{x=0}^{x=2} u^{-\frac{1}{2}}\, du$$

This is now a standard form which we can integrate:

$$\begin{aligned} I &= \tfrac{3}{4}\left[2u^{\frac{1}{2}}\right]_{x=0}^{x=2} \\ &= \tfrac{3}{4}\left[2(1+4x)^{\frac{1}{2}}\right]_{x=0}^{x=2} \\ &= \tfrac{3}{4}\left[\left\{2(1+4\cdot 2)^{\frac{1}{2}}\right\} - \left\{2(1+4\cdot 0)^{\frac{1}{2}}\right\}\right] \\ &= \tfrac{3}{4}\left[6-2\right] \\ &= 3 \end{aligned}$$

(b) The integral we want this time will be

$$V = \pi \int_{x=0}^{x=2} y^2 \, dx$$
$$= \pi \int_{x=0}^{x=2} \frac{9}{1+4x} \, dx$$

Now this integral is one of these

$$\int \frac{f'(x)}{[f(x)]^n} \, dx$$

or at least it will when we write it as

$$V = \frac{9\pi}{4} \int_{x=0}^{x=2} \frac{4}{1+4x} \, dx$$

using that cunning "multiply by 4 and divide by 4" trick. So this will be substiution again, and the substaitution will be the same as in part (a). Consequently, this integral will be

$$V = \frac{9\pi}{4} \int_{x=0}^{x=2} \frac{4}{u} \cdot \frac{1}{4} \, du$$
$$= \frac{9\pi}{4} \int_{x=0}^{x=2} \frac{1}{u} \, du$$

which is a standard form, so we can integrate it:

$$V = \frac{9\pi}{4} \Big[\ln(u) \Big]_{x=0}^{x=2}$$
$$= \frac{9\pi}{4} \Big[\ln(1+4x) \Big]_{x=0}^{x=2}$$
$$= \frac{9\pi}{4} \Big[\ln(1+4\cdot 2) - \ln(1+4\cdot 0) \Big]$$
$$= \frac{9\pi}{4} \Big[\ln(9) - \ln(1) \Big]$$
$$= \frac{9\pi \ln(9)}{4}$$

Tips, Tricks and Patterns

Patterns:

$$\int f'(x) \cdot [f(x)]^n \, dx, \qquad \int f'(x) \cdot e^{f(x)} \, dx, \qquad \int \frac{f'(x)}{[f(x)]^n} \, dx \qquad \text{indeed...} \qquad \int f'(x) \cdot g[f(x)] \, dx$$

Question 6

The Question

(a) Find

$$\int \tan^2(x) \, dx$$

(2 marks)

(b) Use integration by parts to find

$$\int \frac{1}{x^3} \ln(x) \, dx$$

(4 marks)

(c) Use the substitution $u = 1 + e^x$ to show that

$$\int \frac{e^{3x}}{1+e^x} \, dx = \tfrac{1}{2} e^{2x} - e^x + \ln(1+e^x) + k$$

where k is a constant. **(7 marks)**

My Answer

(a) Now, $\tan^2(x)$ isn't one of our standard forms. It won't be partial fractions; could be parts; could be substitution; could be trigonometrical identities. Urgh. What do we do? Actually, trig identities? There is one for $\tan^2(x)$, isn't there? Now what was it again?

You can work this out starting from

$$\sin^2(x) + \cos^2(x) \equiv 1$$

by dividing both sides of the identity by $\cos^2(x)$:

$$\frac{\sin^2(x)}{\cos^2(x)} + \frac{\cos^2(x)}{\cos^2(x)} \equiv \frac{1}{\cos^2(x)}$$
$$\tan^2(x) + 1 \equiv \sec^2(x)$$

Aha! And $\sec^2(x)$ is one of our standard forms! So

$$\begin{aligned} I &= \int \tan^2(x)\, dx \\ &= \int \sec^2(x) - 1\, dx \\ &= \tan(x) - x + C \end{aligned}$$

(b) Parts! Yippee! Here, we don't know how to integrate $\ln(x)$, so we have to put that in the D column: So our integral becomes

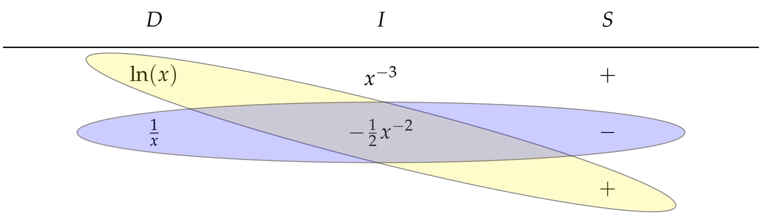

Figure 19: Integrating $\int \frac{1}{x^3} \ln(x)\, dx$

$$\begin{aligned} \int \frac{1}{x^3} \ln(x)\, dx &= -\frac{1}{2} \cdot \frac{1}{x^2} \ln(x) + \int \frac{1}{x} \cdot \frac{1}{2x^2} \ln(x)\, dx \\ &= -\frac{1}{2x^2} \ln(x) + \int \frac{1}{2x^3}\, dx \\ &= -\frac{1}{2x^2} \ln(x) + \int \tfrac{1}{2} x^{-3}\, dx \\ &= -\frac{1}{2x^2} \ln(x) + \tfrac{1}{2} \cdot -\tfrac{1}{2} x^{-2} \\ &= -\frac{1}{2x^2} \ln(x) - \frac{1}{4x^2} + C \end{aligned}$$

(c) here, we are told that we need to use substitution, and which substitution to use. So...

If $\quad u = 1 + e^x \quad$ then $\quad \frac{du}{dx} = e^x \quad$ and $\quad dx = \frac{1}{e^x}\, du$

Shoving this stuff into our integral, and also noticing that the numerator is $e^{3x} = (e^x)^3 = (u-1)^3$,

$$\begin{aligned} \int \frac{e^{3x}}{1+e^x}\, dx &= \int \frac{(e^x)^3}{1+e^x}\, dx \\ &= \int \frac{(u-1)^3}{u} \cdot \frac{1}{e^x}\, du \\ &= \int \frac{(u-1)^3}{u} \cdot \frac{1}{u-1}\, du \\ &= \int \frac{(u-1)^2}{u}\, du \end{aligned}$$

Multiplying out the numerator, we get

$$\begin{aligned}\int \frac{e^{3x}}{1+e^x}\,dx &= \int \frac{u^2-2u+1}{u}\,du \\ &= \int \frac{u^2}{u} - \frac{2u}{u} + \frac{1}{u}\,du \\ &= \int u - 2 + \frac{1}{u}\,du\end{aligned}$$

These are now all standard forms, so we can integrate:

$$\begin{aligned}\int \frac{e^{3x}}{1+e^x}\,dx &= \tfrac{1}{2}u^2 - 2u + \ln(u) + C \\ &= \tfrac{1}{2}(1+e^x)^2 - 2(1+e^x) + \ln(1+e^x) + C \\ &= \tfrac{1}{2}(1+2e^x+e^{2x}) - 2 - 2e^x + \ln(1+e^x) + C \\ &= e^x + \tfrac{1}{2}e^{2x} - 2e^x + \ln(1+e^x) + D \\ &= \tfrac{1}{2}e^{2x} - e^x + \ln(1+e^x) + D\end{aligned}$$

absorbing the $\frac{1}{2}$ and the -2 into the integration constant.

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9 Edexcel C4 June 2009

Question 2

The Question

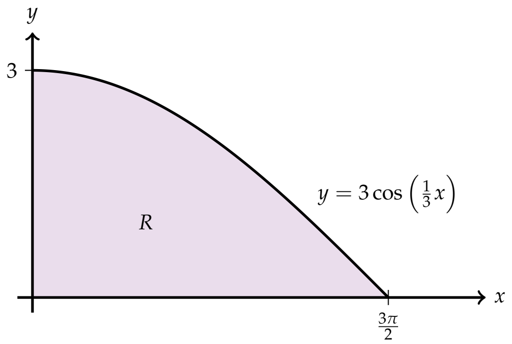

Figure 20: A graph of $y = 3\cos\left(\frac{1}{3}x\right)$

(a) ...

(b) ...

(c) Use integration to find the exact area of R. **(3 marks)**

My Answer

(c) The integral we want will be

$$\int_{x=0}^{x=\frac{3\pi}{2}} 3\cos\left(\frac{1}{3}x\right)\, dx$$

This is very close to a standard form, and we can turn it into one using the substitution

$$\text{If} \quad u = \tfrac{1}{3}x \quad \text{then} \quad \frac{du}{dx} = \frac{1}{3} \quad \text{and} \quad dx = 3\,du$$

So,

$$\begin{aligned}\int_{x=0}^{x=\frac{3\pi}{2}} 3\cos\left(\tfrac{1}{3}x\right)\, dx &= \int_{x=0}^{x=\frac{3\pi}{2}} 3\cos(u)\cdot 3\, du \\ &= 9\int_{x=0}^{x=\frac{3\pi}{2}} \cos(u)\, du\end{aligned}$$

This is now a standard form, so we can integrate it:

$$\begin{aligned}\int_{x=0}^{x=\frac{3\pi}{2}} 3\cos\left(\tfrac{1}{3}x\right)\, dx &= 9\Big[\sin(u)\Big]_{x=0}^{x=\frac{3\pi}{2}} \\ &= 9\Big[\sin\left(\tfrac{1}{3}x\right)\Big]_{x=0}^{x=\frac{3\pi}{2}} \\ &= 9\Big[\Big\{\sin\left(\tfrac{1}{3}\cdot\tfrac{3\pi}{2}\right)\Big\} - \Big\{\sin\left(\tfrac{1}{3}\cdot 0\right)\Big\}\Big] \\ &= 9\Big[\Big\{\sin\left(\tfrac{\pi}{2}\right)\Big\} - \Big\{\sin(0)\Big\}\Big] \\ &= 9\Big[\Big\{1\Big\} - \Big\{0\Big\}\Big] \\ &= 9\end{aligned}$$

Tips, Tricks and Patterns

Tips:

- What standard form is the integrand closest to?

Question 3

The Question

$$f(x) = \frac{4-2x}{(2x+1)(x+1)(x+3)} = \frac{A}{2x+1} + \frac{B}{x+1} + \frac{C}{x+3}$$

(a) Find the values of the constants A, B and C. **(4 marks)**

(b) (i) Hence find

$$\int f(x)\ dx$$

(3 marks)

(ii) Find

$$\int_{x=0}^{x=2} f(x)\ dx$$

in the form $\ln(k)$, where k is a constant. **(3 marks)**

My Answer

(a) Going through the motions, this turns out to be

$$f(x) = \frac{4-2x}{(2x+1)(x+1)(x+3)} = \frac{4}{2x+1} - \frac{3}{x+1} + \frac{1}{x+3}$$

(b) Now **all** partial fractions integrals end up with terms like this

$$\int \frac{a}{bx+c}\ dx$$

to integrate. And there is a standard way that you do these. When you learn about using substitution to solve integrals, there is a very common family of integrals that you learn about. These integrals are

$$\int f'(x) \cdot [f(x)]^n\ dx, \qquad \int f'(x) \cdot e^{f(x)}\ dx, \qquad \int \frac{f'(x)}{[f(x)]^n}\ dx, \qquad \text{indeed...} \qquad \int f'(x) \cdot g[f(x)]\ dx$$

where the coloured bits are the things to look out for. If you have an integral of one of these types, then you use substitution to transform the integral into a standard form, and the substitution is $u = f(x)$.

So, if we write the integral like this

$$\int f(x)\ dx = 2\int \frac{2}{2x+1}\ dx - 3\int \frac{1}{x+1}\ dx + \int \frac{1}{x+3}\ dx$$

and using the fact that

$$\int \frac{f'(x)}{f(x)}\ dx = \ln[f(x)]$$

that we've seen before, then

$$\begin{aligned} \int f(x)\ dx &= 2\ln(2x+1) - 3\ln(x+1) + \ln(x+3) + \ln(C) \\ &= \ln([2x+1]^2) - \ln([x+1]^3) + \ln(x+3) + \ln(C) \\ &= \ln\left(\frac{C[2x+1]^2[x+3]}{[x+1]^3}\right) \end{aligned}$$

using the log rules to combine all the logs into one.

(c) So the integral we want is

$$\int_{x=0}^{x=2} f(x)\,dx = \left[\ln\left(\frac{[2x+1]^2[x+3]}{[x+1]^3}\right)\right]_{x=0}^{x=2}$$
$$= \left[\left\{\ln\left(\frac{[2\cdot 2+1]^2[2+3]}{[2+1]^3}\right)\right\} - \left\{\ln\left(\frac{[2\cdot 0+1]^2[0+3]}{[0+1]^3}\right)\right\}\right]$$
$$= \left[\left\{\ln\left(\frac{5^2\cdot 5}{3^3}\right)\right\} - \{\ln(3)\}\right]$$
$$= \left[\ln\left(\frac{5^3}{3^4}\right)\right]$$
$$= \ln\left(\frac{125}{81}\right)$$

Tips, Tricks and Patterns

Patterns:

$$\int f'(x)\cdot[f(x)]^n\,dx, \quad \int f'(x)\cdot e^{f(x)}\,dx, \quad \int \frac{f'(x)}{[f(x)]^n}\,dx \quad \text{indeed...} \quad \int f'(x)\cdot g[f(x)]\,dx$$

Question 6

The Question

(a) Find

$$\int \sqrt{5-x}\,dx$$

(2 marks)

Figure 21 shows a sketch of the curve with the equation

$$y = (x-1)\sqrt{5-x} \qquad 1 \le x \le 5$$

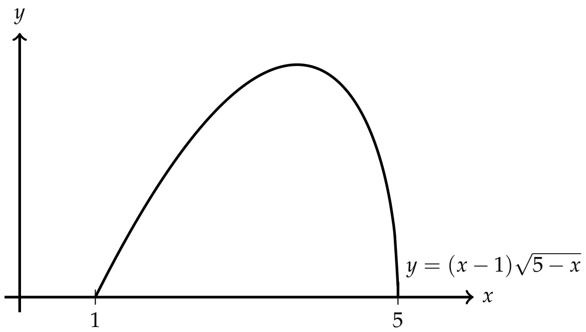

Figure 21: A graph of $y = (x-1)\sqrt{5-x}$

(b) (i) Using integration by parts, or otherwise, find

$$\int (x-1)\sqrt{5-x}\,dx$$

(4 marks)

(ii) Hence find

$$\int_{x=1}^{x=5} (x-1)\sqrt{5-x}\,dx$$

(2 marks)

My Answer

(a) This is probably not partial fractions, trigonometrical identities or parts, so it's likely to be substitution. So what would the substitution be? Perhaps an obvious one to try first would be $u = 5 - x$. Yeah - that woud make the integrand something that looks like $x^{\frac{1}{2}}$! So let's see what happens:

If $\quad u = 5 - x \quad$ then $\quad \frac{du}{dx} = -1 \quad$ and $\quad dx = -du$

Plugging this in, we get

$$\begin{aligned} \int \sqrt{5-x}\, dx &= \int \sqrt{u} \cdot -du \\ &= -\int \sqrt{u}\, du \\ &= -\int u^{\frac{1}{2}}\, du \end{aligned}$$

which is now a standard form, so we can integrate it:

$$\begin{aligned} \int \sqrt{5-x}\, dx &= -\tfrac{2}{3}u^{\frac{3}{2}} + C \\ &= -\tfrac{2}{3}(5-x)^{\frac{3}{2}} + C \end{aligned}$$

(b) (i) Now we are encouraged to use parts here, so let's get DIS out for a spin. What to put in what columns? Well, we've just integrated $\sqrt{5-x}$, so we could put that into the I column. Trouble with that is that we would have to integrate $-\frac{2}{3}(5-x)^{\frac{3}{2}}$. Oh, I reckon that's not so bad: we can follow the lead from part (a): add one to the power, divide by the new power, multiply by -1:

D	I	S
$x-1$	$\sqrt{5-x}$	$+$
1	$-\frac{2}{3}(5-x)^{\frac{3}{2}}$	$-$
0	$\frac{4}{15}(5-x)^{\frac{5}{2}}$	$+$
		$-$

Figure 22: Integrating $\int(x-1)\sqrt{5-x}\,dx$

This would give

$$\int (x-1)\sqrt{5-x}\, dx = -\tfrac{2}{3}(x-1)(5-x)^{\frac{3}{2}} - \tfrac{4}{15}(5-x)^{\frac{5}{2}} + C$$

(c) So

$$\begin{aligned} \int_{x=1}^{x=5} (x-1)\sqrt{5-x}\, dx &= \left[-\tfrac{2}{3}(x-1)(5-x)^{\frac{3}{2}} - \tfrac{4}{15}(5-x)^{\frac{5}{2}}\right]_{x=1}^{x=5} \\ &= \left[\left\{-\tfrac{2}{3}(5-1)(5-5)^{\frac{3}{2}} - \tfrac{4}{15}(5-5)^{\frac{5}{2}}\right\} - \left\{-\tfrac{2}{3}(1-1)(5-1)^{\frac{3}{2}} - \tfrac{4}{15}(5-1)^{\frac{5}{2}}\right\}\right] \\ &= \left[\{0\} - \left\{0 - \tfrac{4}{15}\cdot 4^{\frac{5}{2}}\right\}\right] \\ &= \tfrac{4}{15}\cdot 2^5 \\ &= \tfrac{4}{15}\cdot 32 \\ &= \tfrac{128}{15} \end{aligned}$$

Tips, Tricks and Patterns

Tips:

- What standard form is the integrand closest to?

Question 8

The Question

(a) Using the identity $\cos(2\theta) \equiv 1 - 2\sin^2(\theta)$, find

$$\int \sin^2(\theta)\ d\theta$$

(2 marks)

Figure 23 shows part of the curve C with parametric equations

$$x = \tan(\theta), \qquad y = 2\sin(2\theta), \qquad 0 \le \theta \le \tfrac{\pi}{2}$$

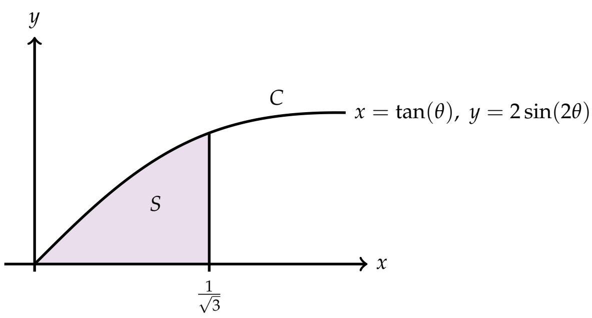

Figure 23: A graph of $x = \tan(\theta),\ y = 2\sin(2\theta)$

The finite shaded region S shown in Figure 23 is bounded by C, the line $x = \frac{1}{\sqrt{3}}$ and the x-axis. The shaded region is rotated through 2π radians about the x-axis to form a solid of revolution.

(b) Show that the volume of the solid of revolution is given by the integral

$$k \int_{x=0}^{x=\frac{\pi}{6}} \sin^2(\theta)\ d\theta$$

where k is a constant. **(5 marks)**

(c) Hence find the exact value for this volume, giving your answer in the form $p\pi^2 + q\pi\sqrt{3}$, where p and q are constants. **(3 marks)**

My Answer

(a) They're telling us how to do this one! Thank you very much! So, if $\cos(2\theta) \equiv 1 - 2\sin^2(\theta)$, then $\sin^2(\theta) = \frac{1}{2}[1 - \cos(2\theta)]$, then

$$\begin{aligned}\int \sin^2(\theta)\ d\theta &= \int \tfrac{1}{2}[1 - \cos(2\theta)]\ d\theta \\ &= \tfrac{1}{2}\int 1 - \cos(2\theta)\ d\theta \\ &= \tfrac{1}{2}\int 1\ d\theta - \tfrac{1}{2}\int \cos(2\theta)\ d\theta\end{aligned}$$

Now technically we would need to do a substitution here for the second integral:

If $\quad u = 2\theta \quad$ then $\quad \dfrac{du}{d\theta} = 2 \quad$ and $\quad d\theta = \frac{1}{2}du$

so that

$$\begin{aligned}\int \sin^2(\theta)\ d\theta &= \tfrac{1}{2}\int 1\ d\theta - \tfrac{1}{2}\int \cos(u)\cdot\tfrac{1}{2}\ du \\ &= \tfrac{1}{2}\int 1\ d\theta - \tfrac{1}{4}\int \cos(u)\ du\end{aligned}$$

which is now a standard form, so we can integrate it

$$\begin{aligned}\int \sin^2(\theta)\ d\theta &= \tfrac{1}{2}\theta - \tfrac{1}{4}\sin(u) + C \\ &= \tfrac{1}{2}\theta - \tfrac{1}{4}\sin(2\theta) + C\end{aligned}$$

(b) The integral we want will be

$$V = \pi \int_{x=0}^{x=\frac{1}{\sqrt{3}}} y^2\ dx$$

Now the y bit is easy. That's just $[2\sin(2\theta)]^2$. But what about the dx? Well, to turn this into $d\theta$, we need to find a relationship between dx and $d\theta$. Well, to do that. we differentiate $x = \tan(\theta)$:

$$\begin{aligned}x &= \tan(\theta) \\ \frac{dx}{d\theta} &= \sec^2(\theta)\end{aligned}$$

That's in the formula book. Now that means that $dx = \sec^2(\theta)\ d\theta$, so our integral becomes

$$V = \pi \int_{x=0}^{x=\frac{1}{\sqrt{3}}} [2\sin(2\theta)]^2 \sec^2(\theta)\ d\theta$$

Now in the answer, there is no 2θ bit. So how can we get rid of that? Well, there's an identity, isn't there: $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$. So:

$$\begin{aligned}V &= \pi \int_{x=0}^{x=\frac{1}{\sqrt{3}}} [2 \cdot 2\sin(\theta)\cos(\theta)]^2 \sec^2(\theta)\ d\theta \\ &= \pi \int_{x=0}^{x=\frac{1}{\sqrt{3}}} [4\sin(\theta)\cos(\theta)]^2 \sec^2(\theta)\ d\theta \\ &= \pi \int_{x=0}^{x=\frac{1}{\sqrt{3}}} 16\sin^2(\theta)\cos^2(\theta)\sec^2(\theta)\ d\theta \\ &= 16\pi \int_{x=0}^{x=\frac{1}{\sqrt{3}}} \sin^2(\theta)\ d\theta\end{aligned}$$

since $\sec(x) = \frac{1}{\cos(x)}$. Now we just have the limits to go. When $x = 0$, then $\tan(\theta) = 0$, which means that $\theta = 0$. And when $x = \frac{1}{\sqrt{3}}$, then $\tan(\theta) = \frac{1}{\sqrt{3}}$, and my trusty calculator tells me that that would mean that $\theta = \frac{\pi}{6}$. So, finally,

$$V = 16\pi \int_{\theta=0}^{\theta=\frac{\pi}{6}} \sin^2(\theta)\ d\theta$$

(c) Puttting parts (a) and (b) together, then, we get

$$\begin{aligned}V &= 16\pi \int_{\theta=0}^{\theta=\frac{\pi}{6}} \sin^2(\theta)\ d\theta \\ &= 16\pi \left[\tfrac{1}{2}\theta - \tfrac{1}{4}\sin(2\theta)\right]_{\theta=0}^{\theta=\frac{\pi}{6}} \\ &= 16\pi \left[\left\{\tfrac{1}{2}\cdot\tfrac{\pi}{6} - \tfrac{1}{4}\sin(2\cdot\tfrac{\pi}{6})\right\} - \left\{\tfrac{1}{2}\cdot 0 - \tfrac{1}{4}\sin(2\cdot 0)\right\}\right] \\ &= 16\pi \left[\left\{\tfrac{\pi}{12} - \tfrac{1}{4}\cdot\tfrac{\sqrt{3}}{2}\right\} - 0\right] \\ &= 16\pi \left[\tfrac{\pi}{12} - \tfrac{\sqrt{3}}{8}\right] \\ &= \tfrac{4}{3}\pi^2 - 2\sqrt{3}\pi\end{aligned}$$

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Question 2

The Question

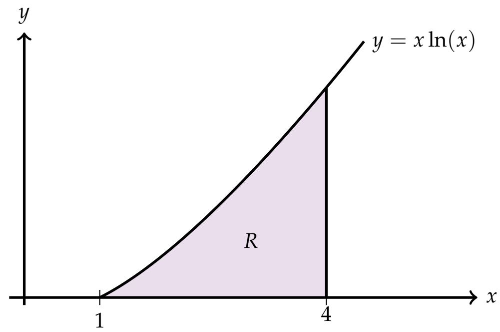

Figure 24: A graph of $y = x\ln(x)$

Figure 24 shows a sketch of the curve with equation $y = x\ln(x)$, $x \geq 1$. The finite region R, shown shaded in Figure 24, is bounded by the curve, the x-axis and the line $x = 4$.

(a) ...

(b) ...

(c) (i) Use integration by parts to find

$$\int x\ln(x)\ dx$$

(4 marks)

(ii) Hence find the exact area of R, giving your answer in the form $\frac{1}{4}\left(a\ln(2)+b\right)$, where a and b are integers. **(3 marks)**

My Answer

(c) (i) Well, this is a classic integration by parts. Whenever we have a polynomial multiplied by a sine, cosine, exponential or log, it's parts. So...DIS is how we do it. As we don't know how to integrate $\ln(x)$, that has to go in the D column:

D	I	S
$\ln(x)$	x	$+$
$\frac{1}{x}$	$\frac{1}{2}x^2$	$-$
		$+$

Figure 25: Integrating $\int x\ln(x)\ dx$

So our integral becomes

$$\begin{aligned}\int x\ln(x)\ dx &= \tfrac{1}{2}x^2\ln(x) - \int \tfrac{1}{x}\cdot\tfrac{1}{2}x^2\ dx \\ &= \tfrac{1}{2}x^2\ln(x) - \tfrac{1}{2}\int x\ dx \\ &= \tfrac{1}{2}x^2\ln(x) - \tfrac{1}{4}x^2 + C\end{aligned}$$

(c) (ii) So the integral we want will be

$$\int_{x=1}^{x=4} x\ln(x)\ dx = \left[\tfrac{1}{2}x^2\ln(x) - \tfrac{1}{4}x^2\right]_{x=1}^{x=4}$$

Plugging the numbers in:

$$\begin{aligned}\int_{x=1}^{x=4} x\ln(x)\,dx &= \left[\left\{\tfrac{1}{2}\cdot 4^2\cdot\ln(4)-\tfrac{1}{4}\cdot 4^2\right\}-\left\{\tfrac{1}{2}\cdot 1^2\cdot\ln(1)-\tfrac{1}{4}\cdot 1^2\right\}\right]\\ &= \left[\{8\ln(4)-4\}-\left\{\tfrac{1}{2}\ln(1)-\tfrac{1}{4}\right\}\right]\\ &= 8\ln(4)-\tfrac{15}{4}\\ &= \tfrac{1}{4}\left[32\ln\left(2^2\right)-15\right]\\ &= \tfrac{1}{4}\left[32\cdot 2\ln(2)-15\right]\\ &= \tfrac{1}{4}\left[64\ln(2)-15\right]\end{aligned}$$

Question 5

The Question

(a) Find

$$\int \frac{9x+6}{x}\,dx, \qquad x>0$$

(2 marks)

(b) Given that $y=8$ at $x=1$, solve the differential equation

$$\frac{dy}{dx}=\frac{(9x+6)\,y^{\frac{1}{3}}}{x}$$

giving your answer in the form $y^2=g(x)$. **(6 marks)**

My Answer

(a) Here we can do the standard thing of just dividing by the denominator:

$$\begin{aligned}\int \frac{9x+6}{x}\,dx &= \int \frac{9x}{x}\,dx+\int \frac{6}{x}\,dx\\ &= \int 9\,dx+6\int \frac{1}{x}\,dx\end{aligned}$$

We now have just standard forms to integrate:

$$\int \frac{9x+6}{x}\,dx = 9x+6\ln(x)+C$$

(b) The only way we know how to solve differential equations is to separate the variables,

$$y^{-\frac{1}{3}}\,dy=\frac{9x+6}{x}\,dx$$

and integrate

$$\int y^{-\frac{1}{3}}\,dy=\int \frac{9x+6}{x}\,dx$$

The left hand side is a standard form; the right hand side we've got from part (a):

$$\tfrac{3}{2}y^{\frac{2}{3}}=9x+6\ln(x)+C$$

To find the C we use the other information we have been given: $y=8$ when $x=1$. So,

$$\begin{aligned}\tfrac{3}{2}\cdot 8^{\frac{2}{3}} &= 9\cdot 1+6\ln(1)+C\\ \tfrac{3}{2}\cdot 2^2 &= 9+0+C\\ \tfrac{3}{2}\cdot 4 &= 9+C\\ 6-9 &= C\\ -3 &= C\end{aligned}$$

So our equation is

$$\begin{aligned}\tfrac{3}{2}y^{\frac{2}{3}} &= 9x+6\ln(x)-3\\ y^{\frac{2}{3}} &= \tfrac{2}{3}(9x+6\ln(x)-3)\\ y^2 &= [6x+4\ln(x)-2]^3\end{aligned}$$

Question 7

The Question

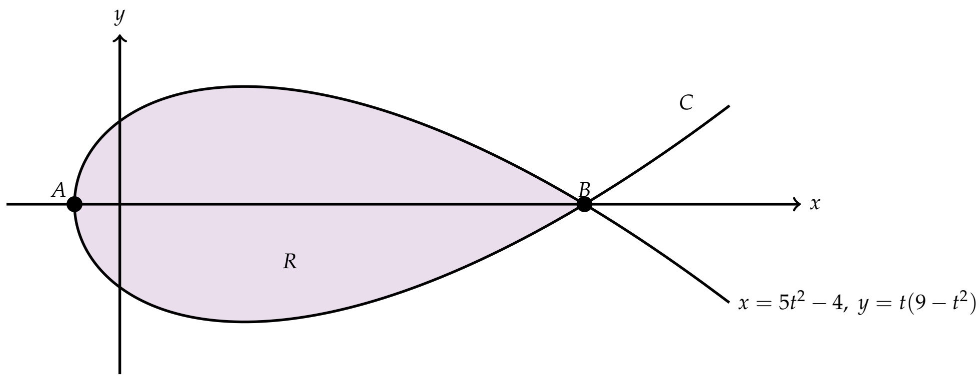

Figure 26: A graph of $x = 5t^2 - 4,\ y = t(9 - t^2)$

Figure 26 shows a sketch of the curve C with parametric equations

$$x = 5t^2 - 4, \qquad y = t(9 - t^2)$$

The curve C cuts the x-axis at the points A and B.

(a) Find the x-coordinate at the point A and the x-coordinate at the point B. **(3 marks)**

The region R, as shown shaded in Figure 26, is enclosed by the loop of the curve.

(b) Use integration to find the area of R. **(6 marks)**

My Answer

(a) When $y = 0$, $t(9 - t^2) = 0$, and so either $t = 0$, or $t^2 = 9$. When $t = 0$, $x = -4$; when $t^2 = 9$, $x = 5 \cdot 9 - 4 = 41$. So A is at $(-4, 0)$, and B is at $(41, 0)$.

(b) the area of R will be the integral

$$R = \int_{x=-4}^{x=41} y\ dx$$

We must try and get this integral to be completely in terms of t. y is easy, that's just $t(9 - t^2)$. To get dx, we differentiate $x = 5t^2 - 4$:

$$\begin{aligned} x &= 5t^2 - 4 \\ \frac{dx}{dt} &= 10t \\ dx &= 10t\ dt \end{aligned}$$

Let's worry about the limits in a minute. So, our integral becomes

$$\begin{aligned} R &= \int_{x=-4}^{x=41} t(9 - t^2)10t\ dt \\ &= 10 \int_{x=-4}^{x=41} t^2(9 - t^2)\ dt \\ &= 10 \int_{x=-4}^{x=41} 9t^2 - t^4\ dt \end{aligned}$$

This is just standard form stuff, so we can integrate it

$$\begin{aligned} R &= 10 \left[3t^3 - \tfrac{1}{5}t^5\right]_{x=-4}^{x=41} \\ &= \left[30t^3 - 2t^5\right]_{x=-4}^{x=41} \end{aligned}$$

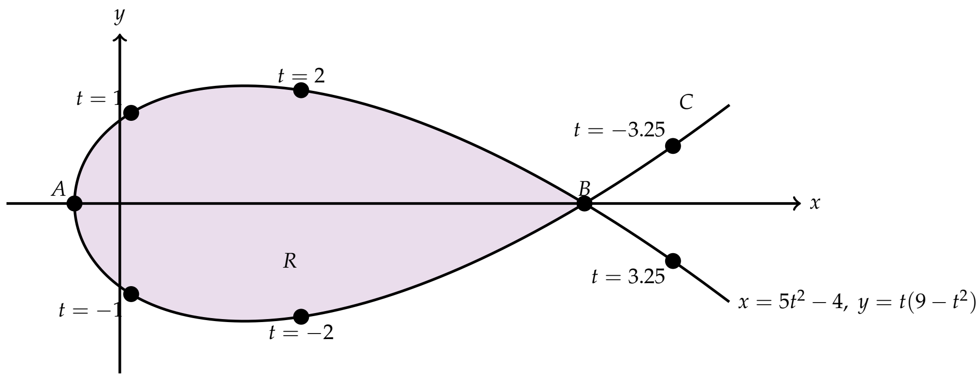

Figure 27: The sweep of the curve

In part (a) we found that the value of t at A was 0, and the values of t at B were ± 3. This means that the curve must sweep down from top right, go clockwise underneath R, through A, over the top of R, through B and down to the right. See Figure 27.

So the values of t at the "beginning and end" of the shaded region R will be $t = -3$ and $t = 3$. So our integral is

$$\begin{aligned} R &= \left[30t^3 - 2t^5\right]_{t=-3}^{t=3} \\ &= \left[\left\{30 \cdot 3^3 - 2 \cdot 3^5\right\} - \left\{30 \cdot (-3)^3 - 2 \cdot (-3)^5\right\}\right] \\ &= \left[\{30 \cdot 27 - 2 \cdot 243\} - \{30 \cdot -27 - 2 \cdot -243\}\right] \\ &= 324 + 324 \\ &= 648 \end{aligned}$$

Question 8

The Question

(a) Using the substitution $x = 2\cos(u)$, or otherwise, find the exact value of

$$I = \int_{x=1}^{x=\sqrt{2}} \frac{1}{x^2\sqrt{4-x^2}}\, dx$$

(7 marks)

Figure 28 shows a sketch of part of the curve with equation $y = \frac{4}{x\left(4-x^2\right)^{\frac{1}{4}}}$, $0 < x < 2$.

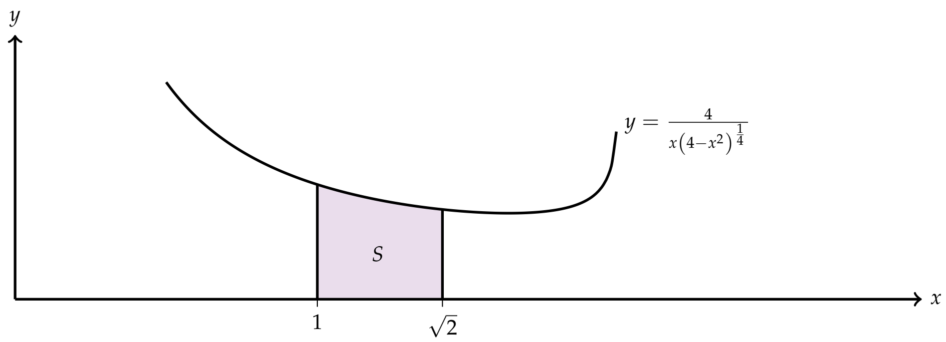

Figure 28: A graph of $y = \frac{4}{x\left(4-x^2\right)^{\frac{1}{4}}}$

The shaded region S, shown in Figure 28, is bounded by the curve, the x-axis and the lines with equations $x = 1$ and $x = \sqrt{2}$. The shaded region S is rotated through 2π radians about the x-axis to form a solid of revolution.

(b) Using your answer to part (a), find the exact volume of the solid of revolution formed. **(3 marks)**

My Answer

(a) So if they tell us what to do we should just go ahead and do it! You have to be *insane* to go down the *or otherwise* route. So, if $x = 2\cos(u)$, then

If $\quad x = 2\cos(u) \quad$ then $\quad \frac{dx}{du} = -2\sin(u) \quad$ and $\quad dx = -2\sin(u)\ du$

Let's plug this in:

$$\begin{aligned} I &= \int_{x=1}^{x=\sqrt{2}} \frac{1}{\left[2\cos(u)\right]^2 \sqrt{4 - \left[2\cos(u)\right]^2}} \cdot -2\sin(u)\ du \\ &= -2 \int_{x=1}^{x=\sqrt{2}} \frac{\sin(u)}{4\cos^2(u)\sqrt{4 - 4\cos^2(u)}}\ du \\ &= -2 \int_{x=1}^{x=\sqrt{2}} \frac{\sin(u)}{4\cos^2(u) \cdot 2\sqrt{1 - \cos^2(u)}}\ du \\ &= -\tfrac{1}{4} \int_{x=1}^{x=\sqrt{2}} \frac{\sin(u)}{\cos^2(u)\sin(u)}\ du \\ &= -\tfrac{1}{4} \int_{x=1}^{x=\sqrt{2}} \sec^2(u)\ du \end{aligned}$$

And we have our standard form. Let's tackle the limits now. When $x = 1$, $u = \frac{\pi}{3}$ (use your calculator), and when $x = \sqrt{2}$, $u = \frac{\pi}{4}$, so our integral becomes

$$\begin{aligned} I &= -\tfrac{1}{4}\Big[\tan(u)\Big]_{u=\frac{\pi}{3}}^{u=\frac{\pi}{4}} \\ &= -\tfrac{1}{4}\left[\left\{\tan\left(\tfrac{\pi}{4}\right)\right\} - \left\{\tan\left(\tfrac{\pi}{3}\right)\right\}\right] \\ &= -\tfrac{1}{4}\left[\{1\} - \left\{\sqrt{3}\right\}\right] \\ &= \tfrac{1}{4}\left[\sqrt{3} - 1\right] \end{aligned}$$

(b) The integral we want will be

$$\begin{aligned} V &= \pi \int_{x=1}^{x=\sqrt{2}} y^2\ dx \\ &= \pi \int_{x=1}^{x=\sqrt{2}} \left(\frac{4}{x\,(4 - x^2)^{\frac{1}{4}}}\right)^2 dx \\ &= 16\pi \int_{x=1}^{x=\sqrt{2}} \frac{1}{x^2\,(4 - x^2)^{\frac{1}{2}}}\ dx \end{aligned}$$

which, using the result from part (a) will be

$$\begin{aligned} V &= 16\pi\tfrac{1}{4}\left[\sqrt{3} - 1\right] \\ &= 4\pi\left[\sqrt{3} - 1\right] \end{aligned}$$

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Question 2

The Question

Using the substitution $u = \cos(x) + 1$, or otherwise, show that

$$I = \int_{x=0}^{x=\frac{\pi}{2}} e^{\cos(x)+1} \sin(x)\ dx = e(e-1)$$

(6 marks)

My Answer

(a) I don't turn down the offer of help: let's use the substitution provided, namely $u = \cos(x) + 1$:

If $\quad \boldsymbol{u = \cos(x) + 1} \quad$ then $\quad \boldsymbol{\frac{du}{dx} = -\sin(x)} \quad$ and $\quad \boldsymbol{dx = -\frac{1}{\sin(x)}\ du}$

Let's plug it in:

$$\begin{aligned} I &= \int_{x=0}^{x=\frac{\pi}{2}} e^u \sin(x) \cdot -\frac{1}{\sin(x)}\ du \\ &= -\int_{x=0}^{x=\frac{\pi}{2}} e^u\ du \end{aligned}$$

Now for the limits: when $x = 0$, $u = \cos(0) + 1 = 2$, and when $x = \frac{\pi}{2}$, $u = \cos\left(\frac{\pi}{2}\right) + 1 = 1$. So...

$$\begin{aligned} I &= -\int_{u=2}^{u=1} e^u\ du \\ &= -\left[e^u\right]_{u=2}^{u=1} \\ &= -\left[e^1 - e^2\right] \\ &= e^2 - e \\ &= e(e-1) \end{aligned}$$

Question 6

The Question

$$f(\theta) = 4\cos^2(\theta) - 3\sin^2(\theta)$$

(a) Show that

$$f(\theta) = \tfrac{1}{2} + \tfrac{7}{2}\cos(2\theta)$$

(3 marks)

(b) Hence, using calculus, find the exact value of

$$\int_{\theta=0}^{\theta=\frac{\pi}{2}} \theta f(\theta)\ d\theta$$

(7 marks)

My Answer

(a) As we have a 2θ thing in the answer, this must be a trigonometrical identity question. Which one? Well, very probably the one with $\cos(2\theta)$ in it! From the addition formulae,

$$\cos(2\theta) \equiv \cos^2(\theta) - \sin^2(\theta) \equiv 2\cos^2(\theta) - 1$$

So how can we use this? Well, we could start by expressing $f(\theta)$ purely in terms of say $\cos^2(\theta)$ using the identity $\sin^2(x) + \cos^2(x) \equiv 1$:

$$\begin{aligned} f(\theta) &= 4\cos^2(\theta) - 3\sin^2(\theta) \\ &= 4\cos^2(\theta) - 3\big[1 - \cos^2(\theta)\big] \\ &= 4\cos^2(\theta) - 3 + 3\cos^2(\theta) \\ &= 7\cos^2(\theta) - 3 \end{aligned}$$

So now we can use our identity for $\cos(2\theta)$:

$$\begin{aligned} f(\theta) &= 7\big[\tfrac{1}{2}\{1 + \cos(2\theta)\}\big] - 3 \\ &= \tfrac{7}{2} + \tfrac{7}{2}\cos(2\theta) - 3 \\ &= \tfrac{1}{2} + \tfrac{7}{2}\cos(2\theta) \end{aligned}$$

(b) So the integral we want will be

$$\begin{aligned} I &= \int_{\theta=0}^{\theta=\frac{\pi}{2}} \theta \cdot \left[\tfrac{1}{2} + \tfrac{7}{2}\cos(2\theta)\right] \, d\theta \\ &= \tfrac{1}{2} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \theta\big[1 + 7\cos(2\theta)\big] \, d\theta \\ &= \tfrac{1}{2} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \theta \, d\theta + \tfrac{7}{2} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \theta\cos(2\theta) \, d\theta \end{aligned}$$

Well, the second integral is a classic integration by parts. Whenever we have a polynomial multiplied by a sine, cosine, exponential or log, it's parts. So...have a look at Figure 29 to see how we do it using DIS:

D	I	S
θ	$\cos(2\theta)$	$+$
1	$\frac{1}{2}\sin(2\theta)$	$-$
0	$-\frac{1}{4}\cos(2\theta)$	$+$
		$-$

Figure 29: Integrating $\int \theta\cos(2\theta) \, d\theta$

So our integrals will be:

$$\begin{aligned} I &= \tfrac{1}{2} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \theta \, d\theta + \tfrac{7}{2} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \theta\cos(2\theta) \, d\theta \\ &= \tfrac{1}{2}\left[\tfrac{1}{2}\theta^2\right]_{\theta=0}^{\theta=\frac{\pi}{2}} + \tfrac{7}{2}\left[\tfrac{1}{2}\theta\sin(2\theta) + \tfrac{1}{4}\cos(2\theta)\right]_{\theta=0}^{\theta=\frac{\pi}{2}} \\ &= \left[\tfrac{1}{4}\theta^2 + \tfrac{7}{4}\theta\sin(2\theta) + \tfrac{7}{8}\cos(2\theta)\right]_{\theta=0}^{\theta=\frac{\pi}{2}} \end{aligned}$$

Now we plug the numbers in:

$$\begin{aligned} I &= \left[\left\{\tfrac{1}{4}\left(\tfrac{\pi}{2}\right)^2 + \tfrac{7}{4} \cdot \tfrac{\pi}{2} \cdot \sin\left(2 \cdot \tfrac{\pi}{2}\right) + \tfrac{7}{8}\cos\left(2 \cdot \tfrac{\pi}{2}\right)\right\} - \left\{\tfrac{1}{4} \cdot 0^2 + \tfrac{7}{4} \cdot 0\sin(2 \cdot 0) + \tfrac{7}{8}\cos(2 \cdot 0)\right\}\right] \\ &= \left[\left\{\tfrac{\pi^2}{16} + 0 - \tfrac{7}{8}\right\} - \left\{0 + 0 + \tfrac{7}{8}\right\}\right] \\ &= \tfrac{\pi^2}{16} - \tfrac{7}{4} \end{aligned}$$

Question 8

The Question

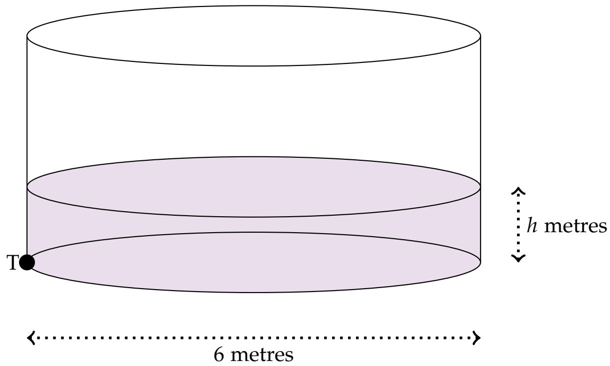

Figure 30: A cylindrical water tank

Figure 30 shows a cylindrical water tank. The diameter of a circular cross-section of the tank is 6m. Water is flowing into the tank at a constant rate of 0.48π m^3 min^{-1}. At time t minutes, the depth of the water in the tank is h metres. There is a tap at a point T at the bottom of the tank. When the tap is open, water leaves the tank at a rate of $0.6\pi h$ m^3 min^{-1}.

(a) Show that t minutes after the tap has been opened

$$75\frac{dh}{dt} = 4 - 5h$$

(5 marks)

When $t = 0$, $h = 0.2$.

(b) Find the value of t when $h = 0.5$. **(6 marks)**

My Answer

(a) ...

(b) The only way we know how to solve a differential equation is to separate the variables

$$\frac{75}{4-5h}\,dh = dt$$

and integrate

$$\int \frac{75}{4-5h}\,dh = \int dt$$

Now I've spotted something here: the left hand side reminds me of that old

$$\int \frac{f'(x)}{f(x)}\,dx = \ln\left[f(x)\right]$$

thing. Remember? So, we need the top of the left hand integrand to be the differential of the bottom:

$$-15\int \frac{-5}{4-5h}\,dh = \int dt$$

using the fact that $-15 \times -5 = +75$. So, we can integrate:

$$-15\ln(4-5h) = t + C$$

Now, when $t = 0$, $h = \frac{1}{5}$, so

$$-15\ln\left(4 - 5\cdot\tfrac{1}{5}\right) = 0 + C$$
$$-15\ln(3) = C$$

So our equation becomes

$$-15\ln(4-5h) = t - 15\ln(3)$$

so that

$$\begin{aligned} t &= 15\ln(3) - 15\ln(4-5h) \\ &= 15\left[\ln(3) - \ln(4-5h)\right] \\ &= 15\left[\ln\left(\frac{3}{4-5h}\right)\right] \end{aligned}$$

so when $h = 0.5$,

$$\begin{aligned} t &= 15\left[\ln\left(\frac{3}{4-5\cdot 0.5}\right)\right] \\ &= 15\left[\ln\left(\frac{3}{4-2.5}\right)\right] \\ &= 15\left[\ln\left(\frac{3}{1.5}\right)\right] \\ &= 15\left[\ln(2)\right] \end{aligned}$$

minutes.

Tips, Tricks and Patterns

Patterns:

$$\int f'(x)\cdot[f(x)]^n\,dx, \qquad \int f'(x)\cdot e^{f(x)}\,dx, \qquad \int \frac{f'(x)}{[f(x)]^n}\,dx \qquad \text{indeed...} \qquad \int f'(x)\cdot g[f(x)]\,dx$$

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Question 1

The Question

Use integration to find the exact value of

$$\int_{x=0}^{x=\frac{\pi}{2}} x\sin(2x)\ dx$$

(6 marks)

My Answer

Well, this is a classic integration by parts. Whenever we have a polynomial multiplied by a sine, cosine, exponential or log, it's parts. So...DIS is how we do it:

D	I	S
x	$\sin(2x)$	$+$
1	$-\frac{1}{2}\cos(2x)$	$-$
0	$-\frac{1}{4}\sin(2x)$	$+$
		$-$

Figure 31: Integrating $\int x\sin(2x)\ dx$

So our integral becomes

$$\begin{aligned}
\int_{x=0}^{x=\frac{\pi}{2}} x\sin(2x)\ dx &= \left[-\tfrac{1}{2}x\cos(2x) + \tfrac{1}{4}\sin(2x)\right]_{x=0}^{x=\frac{\pi}{2}} \\
&= \left[\left\{-\tfrac{1}{2}\cdot\tfrac{\pi}{2}\cdot\cos\left(2\cdot\tfrac{\pi}{2}\right) + \tfrac{1}{4}\sin\left(2\cdot\tfrac{\pi}{2}\right)\right\} - \left\{-\tfrac{1}{2}x\cos(2\cdot 0) + \tfrac{1}{4}\sin(2\cdot 0)\right\}\right] \\
&= \left[\left\{-\tfrac{\pi}{4}\cdot -1 + \tfrac{1}{4}\cdot 0\right\} - \{0+0\}\right] \\
&= \tfrac{\pi}{4}
\end{aligned}$$

Question 3

The Question

(a) Express

$$\frac{5}{(x-1)(3x+2)}$$

in partial fractions. **(3 marks)**

(b) Hence find

$$\int \frac{5}{(x-1)(3x+2)}\ dx, \qquad \text{where } x > 1$$

(3 marks)

(c) Find the particular solution of the differential equation

$$(x-1)(3x+2)\frac{dy}{dx} = 5y, \qquad x > 1$$

for which $y = 8$ at $x = 2$. Give your answer in the form $y = f(x)$. **(6 marks)**

My Answer

(a) This turns out to be

$$\frac{5}{(x-1)(3x+2)} = \frac{1}{x-1} - \frac{3}{3x+2}$$

(b) Using the answer to part (a) we have

$$\begin{aligned}\int \frac{5}{(x-1)(3x+2)}\,dx &= \int \frac{1}{x-1} - \frac{3}{3x+2}\,dx \\ &= \int \frac{1}{x-1}\,dx - \int \frac{3}{3x+2}\,dx\end{aligned}$$

Now **all** partial fractions integrals end up with terms like this

$$\int \frac{a}{bx+c}\,dx$$

to integrate. And there is a standard way that you do these. When you learn about using substitution to solve integrals, there is a very common family of integrals that you learn about. These integrals are

$$\int f'(x)\cdot[f(x)]^n\,dx, \qquad \int f'(x)\cdot e^{f(x)}\,dx, \qquad \int \frac{f'(x)}{[f(x)]^n}\,dx, \qquad \text{indeed...} \qquad \int f'(x)\cdot g[f(x)]\,dx$$

where the coloured bits are the things to look out for. If you have an integral of one of these types, then you use substitution to transform the integral into a standard form, and the substitution is $u = f(x)$.

And when you use this substitution in an integral of the form $\int \frac{f'(x)}{f(x)}\,dx$ you find that

$$\int \frac{f'(x)}{f(x)}\,dx = \ln[f(x)]$$

Now in our inetgrals, we have exactly that form, so we can just write down the answer:

$$\begin{aligned}\int \frac{5}{(x-1)(3x+2)}\,dx &= \ln(x-1) - \ln(3x+2) + \ln(C) \\ &= \ln\left(\frac{C(x-1)}{3x+2}\right)\end{aligned}$$

(c) The only way to solve differential equations at A-Level is to separate out the variables,

$$\frac{1}{y}\,dy = \frac{5}{(x-1)(3x+2)}\,dx$$

and integrate both sides

$$\int \frac{1}{y}\,dy = \int \frac{5}{(x-1)(3x+2)}\,dx$$

Now, from parts (a) and (b) we can integrate this to

$$\ln(y) = \ln\left(\frac{C(x-1)}{3x+2}\right)$$

using the standard form on the left hand side. And so

$$y = \frac{C(x-1)}{3x+2}$$

To find the C we use the fact that $y = 8$ when $x = 2$, so

$$\begin{aligned}8 &= \frac{C(2-1)}{3\cdot 2+2} \\ 8 &= \frac{C}{8} \\ C &= 64\end{aligned}$$

so our solution will be

$$y = \frac{64(x-1)}{3x+2}$$

Tips, Tricks and Patterns

Patterns:

$$\int f'(x) \cdot [f(x)]^n \, dx, \qquad \int f'(x) \cdot e^{f(x)} \, dx, \qquad \int \frac{f'(x)}{[f(x)]^n} \, dx \qquad \text{indeed...} \qquad \int f'(x) \cdot g[f(x)] \, dx$$

Question 6

The Question

The curve C has parametric equations

$$x = \ln(t), \qquad y = t^2 - 2, \qquad t > 0$$

Find

(a) ...

(b) ...

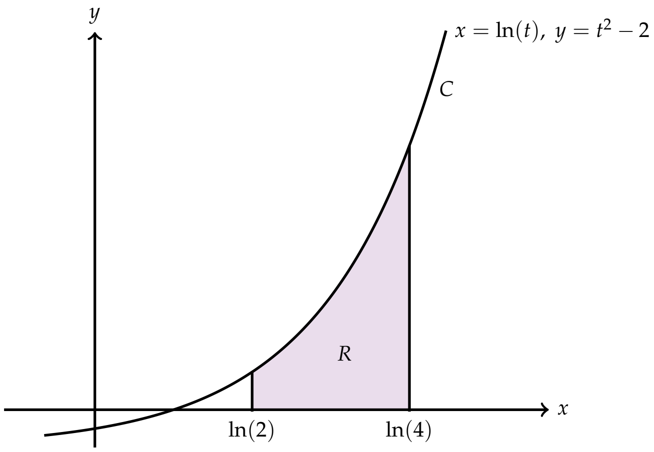

Figure 32: A graph of $x = \ln(t),\ y = t^2 - 2$

The finite area R, shown shaded in Figure 32, is bounded by C, the x-axis, the line $x = \ln(2)$ and the line $x = \ln(4)$. The area R is rotated through 360° about the x-axis.

(c) Use calculus to find the exact volume of the solid generated. **(6 marks)**

My Answer

(c) The integral we want will be

$$V = \pi \int_{x=\ln(2)}^{x=\ln(4)} y^2 \, dx$$

And we have to get everything in terms of t. Now y is easy, that's just going to be $t^2 - 2$. To find dx in terms of dt we need an equation that relates dx and dt. We get that from differentiating the equation $x = \ln(t)$:

$$\begin{aligned} x &= \ln(t) \\ \frac{dx}{dt} &= \frac{1}{t} \\ dx &= \frac{1}{t} \, dt \end{aligned}$$

So, let's plug this in and see what we get:

$$\begin{aligned}
V &= \pi \int_{x=\ln(2)}^{x=\ln(4)} \left(t^2-2\right)^2 \cdot \frac{1}{t}\, dt \\
&= \pi \int_{x=\ln(2)}^{x=\ln(4)} \frac{t^4-4t^2+4}{t}\, dt \\
&= \pi \int_{x=\ln(2)}^{x=\ln(4)} t^3 - 4t + \frac{4}{t}\, dt \\
&= \pi \int_{x=\ln(2)}^{x=\ln(4)} t^3\, dt \quad - \quad 4\pi \int_{x=\ln(2)}^{x=\ln(4)} t\, dt \quad + \quad 4\pi \int_{x=\ln(2)}^{x=\ln(4)} \frac{1}{t}\, dt
\end{aligned}$$

The integrals are now all standard forms, so we just have the limits to worry about. Now, when $x = \ln(2)$, t must be 2, and when $x = \ln(4)$, t must be 4, so

$$V = \pi \int_{t=2}^{t=4} t^3\, dt \quad - \quad 4\pi \int_{t=2}^{t=4} t\, dt \quad + \quad 4\pi \int_{t=2}^{t=4} \frac{1}{t}\, dt$$

Right. Let's integrate:

$$\begin{aligned}
V &= \pi \left[\tfrac{1}{4}t^4\right]_{t=2}^{t=4} - 4\pi \left[\tfrac{1}{2}t^2\right] + 4\pi\Big[\ln(t)\Big]_{t=2}^{t=4} \\
&= 4\pi \left[\tfrac{1}{16}t^4 - \tfrac{1}{2}t^2 + \ln(t)\right]_{t=2}^{t=4} \\
&= 4\pi \left[\left\{\tfrac{1}{16}\cdot 4^4 - \tfrac{1}{2}\cdot 4^2 + \ln(4)\right\} - \left\{\tfrac{1}{16}\cdot 2^4 - \tfrac{1}{2}\cdot 2^2 + \ln(2)\right\}\right] \\
&= 4\pi\left[\{16-8+\ln(4)\} - \{1-2+\ln(2)\}\right] \\
&= 4\pi\left[8+\ln(4)+1-\ln(2)\right] \\
&= 4\pi\left[9+\ln\left(\tfrac{4}{2}\right)\right] \\
&= 4\pi\left[9+\ln(2)\right]
\end{aligned}$$

Question 7

The Question

$$I = \int_{x=2}^{x=5} \frac{1}{4+\sqrt{x-1}}\, dx$$

(a) ...

(b) ...

(c) Using the substitution $x = (u-4)^2 + 1$, or otherwise, and performing the integration, find the exact value of I. **(8 marks)**

My Answer

(a) Well, don't look a gift horse in the mouth: use the hint:

If $\quad x = (u-4)^2+1 \quad$ then $\quad \dfrac{dx}{du} = 2(u-4)\times 1 \quad$ and $\quad dx = 2(u-4)\, du$

Since $\sqrt{x-1} = u - 4$, then we can transform our integral into

$$\begin{aligned} I &= \int_{x=2}^{x=5} \frac{1}{4+u-4} \cdot 2(u-4)\ du \\ &= 2\int_{x=2}^{x=5} \frac{u-4}{u}\ du \\ &= 2\int_{x=2}^{x=5} 1 - \frac{4}{u}\ du \\ &= 2\int_{x=2}^{x=5} 1\ du - 8\int_{x=2}^{x=5} \frac{1}{u}\ du \end{aligned}$$

Both integrals are now standard forms, so we can integrate:

$$\begin{aligned} I &= 2\Big[u\Big]_{x=2}^{x=5} - 8\Big[\ln(u)\Big]_{x=2}^{x=5} \\ &= \Big[2u - 8\ln(u)\Big]_{x=2}^{x=5} \end{aligned}$$

The only thing left now is the limits. Now, when $x = 2$, then $2 = (u-4)^2 + 1$, so $u = 5$, and when $x = 5$, then $5 = (u-4)^2 + 1$, so $u = 6$. Plugging these in:

$$\begin{aligned} I &= \Big[2u - 8\ln(u)\Big]_{u=5}^{u=6} \\ &= \big[\{2\cdot 6 - 8\ln(6)\} - \{2\cdot 5 - 8\ln(5)\}\big] \\ &= \big[\{12 - 8\ln(6)\} - \{10 - 8\ln(5)\}\big] \\ &= 12 - 8\ln(6) - 10 + 8\ln(5) \\ &= 2 + 8[\ln(5) - \ln(6)] \\ &= 2 + 8\left[\ln\left(\tfrac{5}{6}\right)\right] \end{aligned}$$

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Question 4

The Question

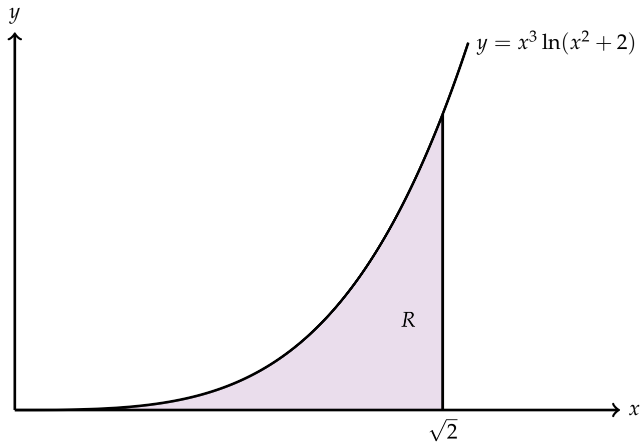

Figure 33: A graph of $y = x^3 \ln(x^2 + 2)$

Figure 33 shows a sketch of the curve with equation $y = x^3 \ln(x^2 + 2)$, $x \geq 0$.

The finite region R, shown shaded in Figure 33, is bounded by the curve, the x-axis and the line $x = \sqrt{2}$.

(a) ...

(b) ...

(c) Use the substitution $u = x^2 + 2$ to show that the area of R is

$$\frac{1}{2} \int_{u=2}^{u=4} (u-2)\ln(u)\ du$$

(4 marks)

(d) Hence, or otherwise, find the exact area of R. **(6 marks)**

My Answer

(c) From the graph, the area we want will be given by

$$I = \int_{x=0}^{x=\sqrt{2}} x^3 \ln(x^2+2)\ dx$$

Again, they tell us that this will be a substitution, and which one to use:

$$\text{If} \qquad u = x^2 + 2 \qquad \text{then} \qquad \frac{du}{dx} = 2x \times 1 \qquad \text{and} \qquad dx = \frac{1}{2x}\ du$$

Plugging this into the integral we get

$$I = \int_{x=0}^{x=\sqrt{2}} x^3 \ln(u) \cdot \frac{1}{2x}\ du$$

and so

$$I = \frac{1}{2} \int_{x=0}^{x=\sqrt{2}} x^2 \ln(u)\ du$$

and since, from the substitution, $x^2 = u - 2$, then

$$I = \frac{1}{2} \int_{x=0}^{x=\sqrt{2}} (u-2)\ln(u)\ du$$

Now we just have the limits to worry about. When $x = 0$, $u = 2$, and when $x = \sqrt{2}$, $u = 4$, so

$$I = \frac{1}{2} \int_{u=2}^{u=4} (u-2)\ln(u)\ du$$

(d) So now we have to integrate this thing. Well, this is a classic integration by parts. Whenever we have a polynomial multiplied by a sine, cosine, exponential or log, it's parts. So...DIS is how we do it. And since we don't know how to integrate $\ln(x)$, that has to go in the D column: So, our integral becomes

D	I	S
$\ln(u)$	$u-2$	$+$
$\frac{1}{u}$	$\frac{1}{2}u^2 - 2u$	$-$
		$+$

Figure 34: Integrating $\int (u-2)\ln(u)\ du$

$$\begin{aligned} I &= \frac{1}{2}\left[\left(\frac{1}{2}u^2 - 2u\right)\ln(u)\right]_{u=2}^{u=4} - \frac{1}{2}\int_{u=2}^{u=4} \frac{1}{u}\cdot\left(\frac{1}{2}u^2 - 2u\right)\ du \\ &= \frac{1}{2}\left[\left(\frac{1}{2}u^2 - 2u\right)\ln(u)\right]_{u=2}^{u=4} - \frac{1}{2}\int_{u=2}^{u=4} \frac{1}{2}u - 2\ du \\ &= \frac{1}{2}\left[\left(\frac{1}{2}u^2 - 2u\right)\ln(u)\right]_{u=2}^{u=4} - \frac{1}{2}\left[\frac{1}{4}u^2 - 2u\right]_{u=2}^{u=4} \\ &= \frac{1}{2}\left[\left(\frac{1}{2}u^2 - 2u\right)\ln(u) - \frac{1}{4}u^2 + 2u\right]_{u=2}^{u=4} \end{aligned}$$

Now we plug the numbers in...

$$\begin{aligned} I &= \frac{1}{2}\left[\left\{\left(\frac{1}{2}4^2 - 2\cdot 4\right)\ln(4) - \frac{1}{4}4^2 + 2\cdot 4\right\} - \left\{\left(\frac{1}{2}2^2 - 2\cdot 2\right)\ln(2) - \frac{1}{4}2^2 + 2\cdot 2\right\}\right] \\ &= \frac{1}{2}\left[\{(8-8)\ln(4) - 4 + 8\} - \{(2-4)\ln(2) - 1 + 4\}\right] \\ &= \frac{1}{2}\left[4 + 2\ln(2) - 3\right] \\ &= \frac{1}{2}\left[1 + 2\ln(2)\right] \end{aligned}$$

Question 7

The Question

Figure 35 shows part of the curve C with parametric equations

$$x = \tan(\theta), \qquad y = \sin(\theta), \qquad 0 \le \theta < \frac{\pi}{2}$$

(a) ...

(b) ...

The finite shaded region S shown in Figure 35 is bounded by the curve C, the line $x = \sqrt{3}$ and the x-axis. This shaded region is rotated through 2π radians about the x-axis to form a solid of revolution.

(c) Find the volume of the solid of revolution, giving your answer in the form $p\pi\sqrt{3} + q\pi^2$, where p and q are constants. **(7 marks)**

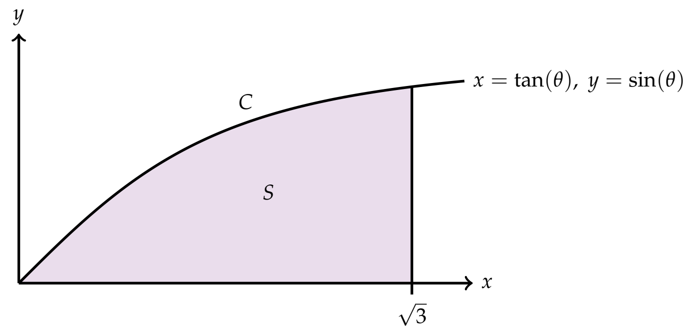

Figure 35: A graph of $x = \tan(\theta),\ y = \sin(\theta)$

My Answer

(c) The integral we need to find will be

$$V = \pi \int_{x=0}^{x=\sqrt{3}} y^2\ dx$$

And we need to convert everything in the integral into θ stuff. The y is easy: that's just $\sin(\theta)$. To find dx we need to find a link between dx and $d\theta$. And to do that we differentiate $x = \tan(\theta)$:

$$\begin{aligned} x &= \tan(\theta) \\ \frac{dx}{d\theta} &= \sec^2(\theta) \\ dx &= \sec^2(\theta)\ d\theta \end{aligned}$$

OK, let's shove all this stuff into the integral:

$$\begin{aligned} V &= \pi \int_{x=0}^{x=\sqrt{3}} \sin^2(\theta) \cdot \sec^2(\theta)\ d\theta \\ &= \pi \int_{x=0}^{x=\sqrt{3}} \tan^2(\theta)\ d\theta \end{aligned}$$

Finally, let's have a look at the limits. When $x = 0$, $\theta = 0$, and when $x = \sqrt{3}$, $\theta = \frac{\pi}{3}$ (use your calculator!). So,

$$V = \pi \int_{\theta=0}^{\theta=\frac{\pi}{3}} \tan^2(\theta)\ d\theta$$

Now, how are we going to integrate $\tan^2(\theta)$. It's not a standard form, so we're going to have to transform it. It won't be partial fractions. It's not likely to be parts. That leaves substitution and trigonometrical identities. Do we know an identity with $\tan^2(\theta)$ in it? Yes we do!! It's

$$\tan^2(\theta) + 1 \equiv \sec^2(\theta)$$

Aha! $\sec^2(\theta)$ is a standard for for integration!! So, here we go...

$$\begin{aligned} V &= \pi \int_{\theta=0}^{\theta=\frac{\pi}{3}} \sec^2(\theta) - 1\ d\theta \\ &= \pi \left[\tan(\theta) - \theta\right]_{\theta=0}^{\theta=\frac{\pi}{3}} \\ &= \pi \left[\left\{\tan\left(\frac{\pi}{3}\right) - \frac{\pi}{3}\right\} - \{\tan(0) - 0\}\right] \\ &= \pi \left[\left\{\sqrt{3} - \frac{\pi}{3}\right\} - \{0\}\right] \\ &= \pi\sqrt{3} - \frac{\pi^2}{3} \end{aligned}$$

Question 8

The Question

(a) FInd

$$\int (4y+3)^{-\frac{1}{2}} \, dy$$

(2 marks)

(b) Given that $y = 1.5$ at $x = -2$, solve the differential equation

$$\frac{dy}{dx} = \frac{\sqrt{4y+3}}{x^2}$$

giving your answer in the form $y = f(x)$. **(6 marks)**

My Answer

(a) The integrand here looks a bit like $x^{-\frac{1}{2}}$, which is a standard form. So, let's try

$$\text{If} \qquad u = 4y+3 \qquad \text{then} \qquad \frac{du}{dy} = 4 \qquad \text{and} \qquad dy = \frac{1}{4}\, du$$

Plugging this into our integral we get

$$\begin{aligned} \int (4y+3)^{-\frac{1}{2}} \, dy &= \int u^{-\frac{1}{2}} \cdot \tfrac{1}{4} \, du \\ &= \tfrac{1}{4} \int (u)^{-\frac{1}{2}} \, du \end{aligned}$$

which is now a standard form, so we can integrate it:

$$\begin{aligned} \int (4y+3)^{-\frac{1}{2}} \, dy &= \tfrac{1}{4} \cdot 2u^{\frac{1}{2}} + C \\ &= \tfrac{1}{2}(4y+3)^{\frac{1}{2}} + C \end{aligned}$$

(b) The only way to solve differential equations at A-Level is to separate the variables

$$\frac{1}{\sqrt{4y+3}} \, dy = \frac{1}{x^2} \, dx$$

and integrate both sides

$$\int \frac{1}{\sqrt{4y+3}} \, dy = \int \frac{1}{x^2} \, dx$$

Now the left hand side we've just found in part (a), and the right hand side is a standard form (x^{-2}). So we can just go ahead and integrate this thing:

$$\tfrac{1}{2}(4y+3)^{\frac{1}{2}} = -\frac{1}{x} + C$$

To find the C we use the fact that $y = 1.5$ when $x = -2$, so

$$\begin{aligned} \tfrac{1}{2}(4 \cdot 1.5 + 3)^{\frac{1}{2}} &= -\frac{1}{-2} + C \\ \tfrac{1}{2}(9)^{\frac{1}{2}} &= \tfrac{1}{2} + C \\ 1 &= C \end{aligned}$$

So, our solution will be

$$\begin{aligned} \tfrac{1}{2}(4y+3)^{\frac{1}{2}} &= -\tfrac{1}{x} + 1 \\ (4y+3)^{\frac{1}{2}} &= 2 - \tfrac{2}{x} \\ 4y+3 &= \left[2 - \tfrac{2}{x}\right]^2 \\ 4y &= \left[2 - \tfrac{2}{x}\right]^2 - 3 \\ y &= \tfrac{1}{4}\left\{\left[2 - \tfrac{2}{x}\right]^2 - 3\right\} \end{aligned}$$

Tips, Tricks and Patterns

Tips:

- What standard form is our integrand closest to?

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Question 2

The Question

(a) Use integration by parts to find

$$\int x \sin(3x)\ dx$$

(3 marks)

(b) Using your answer to part (a), find

$$\int x^2 \cos(3x)\ dx$$

(3 marks)

My Answer

(a) Well, this is a classic integration by parts. Whenever we have a polynomial multiplied by a sine, cosine, exponential or log, it's parts. So...see Figure 36 for DIS is how we do it:

D	I	S
x	$\sin(3x)$	$+$
1	$-\frac{1}{3}\cos(3x)$	$-$
0	$-\frac{1}{9}\sin(3x)$	$+$
		$-$

Figure 36: Integrating $\int x \sin(3x)\ dx$

So our result will be

$$\int x \sin(3x)\ dx = -\tfrac{1}{3}x\cos(3x) + \tfrac{1}{9}\sin(3x) + C$$

(b) Well, this is a classic integration by parts. Whenever we have a polynomial multiplied by a sine, cosine, exponential or log, it's parts. So...see Figure 37 for DIS is how we do it!

D	I	S
x^2	$\cos(3x)$	$+$
$2x$	$\frac{1}{3}\sin(3x)$	$-$
2	$-\frac{1}{9}\cos(3x)$	$+$
0	$-\frac{1}{27}\sin(3x)$	$-$
		$+$

Figure 37: Integrating $\int x^2 \cos(3x)\ dx$

So our result will be

$$\int x^2 \cos(3x)\ dx = \tfrac{1}{3}x^2\sin(3x) + \tfrac{2}{9}x\cos(3x) - \tfrac{2}{27}\sin(3x) + C$$

Question 4

The Question

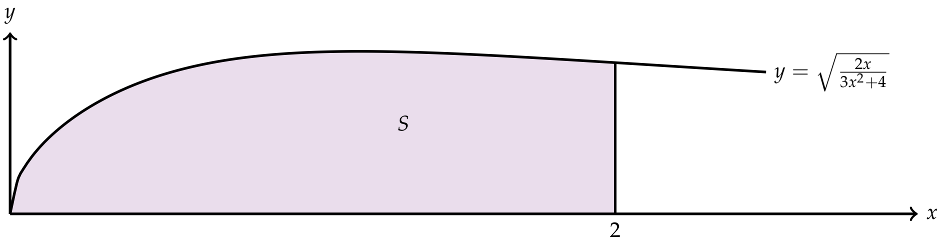

Figure 38: A graph of $y = \sqrt{\frac{2x}{3x^2+4}}$

Figure 38 shows the curve with equation

$$y = \sqrt{\frac{2x}{3x^2+4}}, \qquad x \geq 0$$

The finite region S, shown shaded in Figure 38, is bounded by the curve, the x-axis and the line $x = 2$.

The region S is rotated $360°$ about the x-axis.

Use integration to find the exact value of the volume of the solid generated, giving your answer in the form $k\ln(a)$, where k and a are constants. **(5 marks)**

My Answer

Well, the integral we want this time will be

$$\begin{aligned} V &= \pi \int_{x=0}^{x=2} y^2 \, dx \\ &= \pi \int_{x=0}^{x=2} \left(\sqrt{\frac{2x}{3x^2+4}}\right)^2 dx \\ &= \pi \int_{x=0}^{x=2} \frac{2x}{3x^2+4} \, dx \end{aligned}$$

Oh hello! I've spotted something: the top of this integrand is the differential of the bottom!! Well, almost...and we can make it so by multiplying the inside of the integral by 3 and dividing the outside of the integral by 3:

$$V = \frac{\pi}{3} \int_{x=0}^{x=2} \frac{6x}{3x^2+4} \, dx$$

Why is that good? It's because we have seen elsewhere in this document that $\int \frac{f'(x)}{f(x)} \, dx = \ln\left[f(x)\right]$. So we can just write the answer down:

$$V = \frac{\pi}{3}\left[\ln\left(3x^2+4\right)\right]_{x=0}^{x=2}$$

And now it's just a question of plugging the numbers in...

$$\begin{aligned} V &= \frac{\pi}{3}\left[\left\{\ln\left(3 \cdot 2^2 + 4\right)\right\} - \left\{\ln\left(3 \cdot 0^2 + 4\right)\right\}\right] \\ &= \frac{\pi}{3}\left[\ln(16) - \ln(4)\right] \\ &= \frac{\pi}{3}\ln\left(\frac{16}{4}\right) \\ &= \frac{\pi}{3}\ln(4) \end{aligned}$$

Tips, Tricks and Patterns

Patterns:

$$\int f'(x) \cdot \left[f(x)\right]^n dx, \qquad \int f'(x) \cdot e^{f(x)} \, dx, \qquad \int \frac{f'(x)}{\left[f(x)\right]^n} \, dx \qquad \text{indeed...} \qquad \int f'(x) \cdot g\left[f(x)\right] dx$$

Question 6

The Question

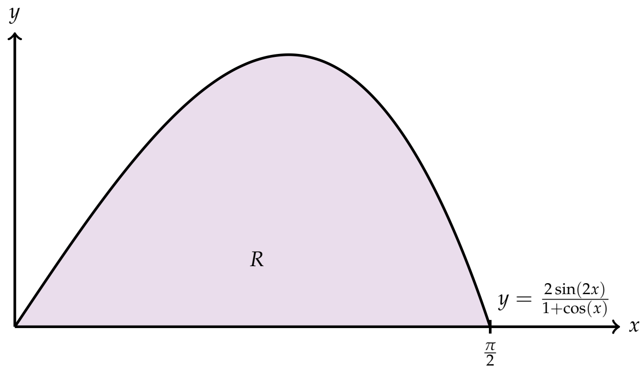

Figure 39: A graph of $y = \frac{2\sin(2x)}{1+\cos(x)}$

Figure 39 shows a sketch of the curve with equation

$$y = \frac{2\sin(2x)}{1+\cos(x)}, \qquad 0 \leq x \leq \frac{\pi}{2}$$

The finite region R, shown shaded in Figure 39, is bounded by the curve and the x-axis.

(a) ...

(b) ...

(c) Using the substitution $u = 1 + \cos(x)$, or otherwise, show that

$$\int \frac{2\sin(2x)}{1+\cos(x)}\, dx = 4\ln[1+\cos(x)] - 4\cos(x) + k$$

where k is a constant. **(5 marks)**

(d) ...

My Answer

(a) Well, this integral will be

$$R = \int_{x=0}^{x=\frac{\pi}{2}} \frac{2\sin(2x)}{1+\cos(x)}\, dx$$

and we are told to use the substitution $u = 1 + \cos(x)$. So let's get on with it...

$$\text{If} \qquad u = 1 + \cos(x) \qquad \text{then} \qquad \frac{du}{dx} = -\sin(x) \qquad \text{and} \qquad dx = -\frac{1}{\sin(x)}\, du$$

The next step is to plug this stuff into the integral:

$$R = \int_{x=0}^{x=\frac{\pi}{2}} \frac{2\sin(2x)}{u} \cdot -\frac{1}{\sin(x)}\, du$$

Now what? Well, on the top of this integrand we've got $\sin(2x)$, and on the bottom there's $\sin(x)$. It would be nice if they cancelled. But we can't do that because one is a $2x$ and the other is an x. So what can we do? Ah! We could use the double angle formula $\sin(2x) = 2\sin(x)\cos(x)$! This would give us

$$R = \int_{x=0}^{x=\frac{\pi}{2}} \frac{2 \cdot 2\sin(x)\cos(x)}{u} \cdot -\frac{1}{\sin(x)}\, du$$

And now the $\sin(x)$ on top and bottom does cancel! Yippee! Tidying things up a bit then, we get

$$R = -4\int_{x=0}^{x=\frac{\pi}{2}} \frac{\cos(x)}{u}\, du$$

Now what can we do with the $\cos(x)$? Ah - we could go back to our substitution: $\cos(x) = u - 1$:

$$R = -4 \int_{x=0}^{x=\frac{\pi}{2}} \frac{u-1}{u} \, du$$
$$= -4 \int_{x=0}^{x=\frac{\pi}{2}} 1 - \frac{1}{u} \, du$$

using the way fractions add/subtract to separate the two terms. Now these are both standard forms, so

$$R = -4\Big[u - \ln(u)\Big]_{x=0}^{x=\frac{\pi}{2}}$$

Now we could convert the limits: when $x = 0$, $u = 2$, and when $x = \frac{\pi}{2}$, $u = 1$. So,

$$R = -4\Big[u - \ln(u)\Big]_{u=2}^{u=1}$$
$$= 4\Big[u - \ln(u)\Big]_{u=1}^{u=2}$$

using that old "multiplying by -1 is equivalent to swapping the limits" ploy. So

$$R = 4\left[\{2 - \ln(2)\} - \{1 - \ln(1)\}\right]$$
$$= 4\left[2 - \ln(2) - 1 + 0\right]$$
$$= 4\left[1 - \ln(2)\right]$$

Question 8

The Question

(a) Express

$$\frac{1}{P(5-P)}$$

in partial fractions. **(3 marks)**

A team of conservationists is studying the population of meerkats on a nature reserve. The population is modelled by the differential equation

$$\frac{dP}{dt} = \frac{1}{15}P(5-P), \qquad t \geq 0$$

where P, in thousands, is the population of meerkats and t is the time measured in years since the study began.

Given that when $t = 0$, $P = 1$,

(b) solve the differential equation, giving your answer in the form

$$P = \frac{a}{b + ce^{-\frac{1}{3}t}}$$

where a, b and c are integers. **(8 marks)**

(c) Hence show that the population cannot exceed 5000. **(1 mark)**

My Answer

(a) This turns out to be

$$\frac{1}{P(5-P)} = \frac{1}{5}\left[\frac{1}{P} + \frac{1}{5-P}\right]$$

(b) I don't know if I've said this already, but the only way we know how to solve differential equations at A-Level is to separate the variables

$$\frac{1}{P(5-P)} \, dP = \frac{1}{15} \, dt$$

and integrate

$$\int \frac{1}{P(5-P)} \, dP = \int \frac{1}{15} \, dt$$

Now from part (a) we can write this as

$$\frac{1}{5}\int \frac{1}{P} + \frac{1}{5-P}\, dP = \frac{1}{15}\int 1\, dt$$

or, indeed

$$\frac{1}{5}\int \frac{1}{P}\, dP + \frac{1}{5}\int \frac{1}{5-P}\, dP = \frac{1}{15}\int 1\, dt$$

Now we could multiply both sides by 15 to get rid of the annoying fractions,

$$3\int \frac{1}{P}\, dP + 3\int \frac{1}{5-P}\, dP = \int 1\, dt$$

and I'm going to use a bit of a trick:

$$3\int \frac{1}{P}\, dP - 3\int \frac{-1}{5-P}\, dP = \int 1\, dt$$

Why? Because **all** partial fractions integrals end up with terms like this

$$\int \frac{a}{bx+c}\, dx$$

to integrate. And there is a standard way that you do these. When you learn about using substitution to solve integrals, there is a very common family of integrals that you learn about. These integrals are

$$\int f'(x)\cdot [f(x)]^n\, dx, \qquad \int f'(x)\cdot e^{f(x)}\, dx, \qquad \int \frac{f'(x)}{[f(x)]^n}\, dx, \qquad \text{indeed...} \qquad \int f'(x)\cdot g[f(x)]\, dx$$

where the coloured bits are the things to look out for. If you have an integral of one of these types, then you use substitution to transform the integral into a standard form, and the substitution is $u = f(x)$.

And if you have been paying attention as you have read through this document, you will have spotted that on several occasions we have used the idea that

$$\int \frac{f'(x)}{f(x)}\, dx = \ln[f(x)]$$

so we can now integrate our differential equation:

$$3\ln(P) - 3\ln(5-P) = t + C$$
$$3\ln\left(\frac{P}{5-P}\right) = t + C$$

To find the C we use the fact that when $t = 0$, $P = 1$. So

$$3\ln\left(\tfrac{1}{4}\right) = C$$

Throwing this in:

$$3\ln\left(\frac{P}{5-P}\right) = t + 3\ln\left(\tfrac{1}{4}\right)$$
$$3\ln\left(\frac{P}{5-P}\right) - 3\ln\left(\tfrac{1}{4}\right) = t$$
$$3\ln\left(\frac{P}{5-P}\right) + 3\ln(4) = t$$

using log rules, so that

$$3\ln\left(\frac{4P}{5-P}\right) = t$$

Now we have to make P the subject of this equation...

$$\ln\left(\frac{4P}{5-P}\right) = \tfrac{1}{3}t$$
$$\frac{4P}{5-P} = e^{\frac{1}{3}t}$$
$$4P = (5-P)e^{\frac{1}{3}t}$$
$$4P = 5e^{\frac{1}{3}t} - Pe^{\frac{1}{3}t}$$
$$4P + Pe^{\frac{1}{3}t} = 5e^{\frac{1}{3}t}$$
$$P\left[4 + e^{\frac{1}{3}t}\right] = 5e^{\frac{1}{3}t}$$

and finally(!)

$$P = \frac{5e^{\frac{1}{3}t}}{4 + e^{\frac{1}{3}t}}$$

(d) To do this bit, we have to let t get very large. Now when t gets very big, $e^{\frac{1}{3}t} >> 4$, so P will approach

$$\begin{aligned} P &= \frac{5e^{\frac{1}{3}t}}{e^{\frac{1}{3}t}} \\ &= \frac{5}{1} \\ &= 5 \end{aligned}$$

but can never get bigger than 5 because the bottom of this fraction is always slightly bigger than $e^{\frac{1}{3}t}$.

Tips, Tricks and Patterns

Patterns:

$$\int f'(x) \cdot [f(x)]^n \, dx, \qquad \int f'(x) \cdot e^{f(x)} \, dx, \qquad \int \frac{f'(x)}{[f(x)]^n} \, dx \qquad \text{indeed...} \qquad \int f'(x) \cdot g[f(x)] \, dx$$

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Question 1

The Question

$$f(x) = \frac{1}{x(3x-1)^2} = \frac{A}{x} + \frac{B}{3x-1} + \frac{C}{(3x-1)^2}$$

(a) Find the values of the constants A, B and C. **(4 marks)**

(b) (i) Hence find

$$\int f(x)\ dx$$

(ii) Find

$$\int_{x=1}^{x=2} f(x)\ dx$$

leaving your answer in the form $a + \ln(b)$, where a and b are constants. **(6 marks)**

My Answer

(a) This turns out to be

$$f(x) = \frac{1}{x(3x-1)^2} = \frac{1}{x} - \frac{3}{3x-1} + \frac{3}{(3x-1)^2}$$

(b) (i) So the integral we want will be

$$\begin{aligned} \int f(x)\ dx &= \int \frac{1}{x(3x-1)^2}\ dx \\ &= \int \frac{1}{x} - \frac{3}{3x-1} + \frac{3}{(3x-1)^2}\ dx \end{aligned}$$

using the result from part (a). Writing these as three separate integrals,

$$\int f(x)\ dx = \int \frac{1}{x}\ dx - \int \frac{3}{3x-1}\ dx + \int \frac{3}{(3x-1)^2}\ dx$$

Are you on the lookout for this

$$\int \frac{f'(x)}{f(x)}\ dx = \ln\left[f(x)\right]$$

yet? I hope so! That accounts for the first two integrals:

$$\int f(x)\ dx = \ln(x) - \ln(3x-1) + \int \frac{3}{(3x-1)^2}\ dx$$

Now, what about the third one? This time the top isn't the differential of the bottom. So how do we do this one? Well, what standard form is the integrand closest to? It looks a bit like x^{-2} to me! That gives me the idea for a substitution:

If $\quad u = 3x - 1 \quad$ then $\quad \frac{du}{dx} = 3 \quad$ and $\quad dx = \frac{1}{3}\ du$

So...

$$\begin{aligned} \int f(x)\ dx &= \ln(x) - \ln(3x-1) + \int \frac{3}{u^2} \cdot \frac{1}{3}\ du \\ &= \ln(x) - \ln(3x-1) + \int u^{-2}\ du \end{aligned}$$

This is now a standard form, so we can integrate it...

$$\begin{aligned} \int f(x)\ dx &= \ln(x) - \ln(3x-1) - \frac{1}{u} + C \\ &= \ln(x) - \ln(3x-1) - \frac{1}{3x-1} + C \end{aligned}$$

(ii) So here we want to evaluate

$$\begin{aligned}\int_{x=1}^{x=2} f(x)\,dx &= \left[\ln(x) - \ln(3x-1) - \frac{1}{3x-1}\right]_{x=1}^{x=2} \\ &= \left[\left\{\ln(2) - \ln(3\cdot 2 - 1) - \frac{1}{3\cdot 2-1}\right\} - \left\{\ln(1) - \ln(3\cdot 1 - 1) - \frac{1}{3\cdot 1 - 1}\right\}\right] \\ &= \ln(2) - \ln(5) - \frac{1}{5} + \ln(2) + \frac{1}{2} \\ &= \ln\left(\frac{2\cdot 2}{5}\right) + \frac{3}{10} \\ &= \ln\left(\frac{4}{5}\right) + \frac{3}{10}\end{aligned}$$

Tips, Tricks and Patterns

Patterns:

$$\int f'(x)\cdot[f(x)]^n\,dx, \qquad \int f'(x)\cdot e^{f(x)}\,dx, \qquad \int \frac{f'(x)}{[f(x)]^n}\,dx \qquad \text{indeed...} \qquad \int f'(x)\cdot g[f(x)]\,dx$$

Question 4

The Question

Given that $y = 2$ at $x = \frac{\pi}{4}$, solve the differential equation

$$\frac{dy}{dx} = \frac{3}{y\cos^2(x)}$$

(5 marks)

My Answer

The only way we can solve differential equations at A-Level is to seaparate the variables

$$\begin{aligned} y\,dy &= \frac{3}{\cos^2(x)}\,dx \\ &= 3\sec^2(x)\,dx\end{aligned}$$

and integrate both sides

$$\int y\,dy = 3\int \sec^2(x)\,dx$$

Well, both sides are standard forms, so

$$\tfrac{1}{2}y^2 = 3\tan(x) + C$$

To find the C we use the fact that $y = 2$ at $x = \frac{\pi}{4}$, so

$$\begin{aligned}\tfrac{1}{2}\cdot 2^2 &= 3\tan\left(\tfrac{\pi}{4}\right) + C \\ 2 &= 3\cdot 1 + C \\ -1 &= C\end{aligned}$$

So our solution is

$$\begin{aligned}\tfrac{1}{2}y^2 &= 3\tan(x) - 1 \\ y^2 &= 6\tan(x) - 2\end{aligned}$$

Question 7

The Question

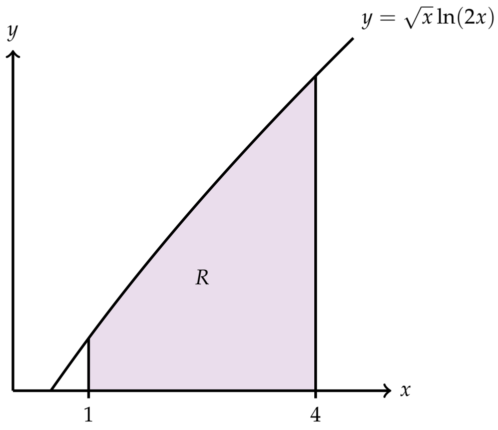

Figure 40: A graph of $y = \sqrt{x}\ln(2x)$

Figure 40 shows a sketch of part of the curve with equation $y = \sqrt{x}\ln(2x)$. The finite region R, shown shaded in Figure 40, is bounded by the curve, the x-axis and the lines $x = 1$ and $x = 4$.

(a) ...

(b) Find

$$\int \sqrt{x}\ln(2x)\ dx$$

(4 marks)

(c) Hence find the exact area of R, giving your answer in the form $a\ln(2) + b$, where a and b are exact constants. **(3 marks)**

My Answer

(b) Well, this is a classic integration by parts. Whenever we have a polynomial multiplied by a sine, cosine, exponential or log, it's parts. Now this is one of those occasions where we have to put the $\ln(2x)$ in the D column because we don't know how to integrate it. So...DIS is how we do it:

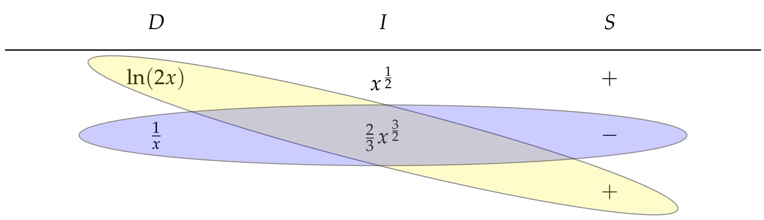

Figure 41: Integrating $\int \sqrt{x}\ln(2x)\ dx$

So our integral becomes

$$\begin{aligned}\int \sqrt{x}\ln(2x)\ dx &= \tfrac{2}{3}x^{\frac{3}{2}}\ln(2x) - \int \frac{1}{x}\cdot\frac{2}{3}x^{\frac{3}{2}}\ dx \\ &= \tfrac{2}{3}x^{\frac{3}{2}}\ln(2x) - \tfrac{2}{3}\int x^{\frac{1}{2}}\ dx \\ &= \tfrac{2}{3}x^{\frac{3}{2}}\ln(2x) - \tfrac{2}{3}\cdot\tfrac{2}{3}x^{\frac{3}{2}} + C \\ &= \tfrac{2}{3}x^{\frac{3}{2}}\ln(2x) - \tfrac{4}{9}x^{\frac{3}{2}} + C\end{aligned}$$

(c) This will be

$$
\begin{aligned}
\int_{x=1}^{x=4} \sqrt{x}\ln(2x)\ dx &= \left[\tfrac{2}{3}x^{\frac{3}{2}}\ln(2x) - \tfrac{4}{9}x^{\frac{3}{2}}\right]_{x=1}^{x=4} \\
&= \left[\left\{\tfrac{2}{3}\cdot 4^{\frac{3}{2}}\cdot\ln(2\cdot 4) - \tfrac{4}{9}\cdot 4^{\frac{3}{2}}\right\} - \left\{\tfrac{2}{3}\cdot 1^{\frac{3}{2}}\cdot\ln(2\cdot 1) - \tfrac{4}{9}\cdot 1^{\frac{3}{2}}\right\}\right] \\
&= \left[\left\{\tfrac{16}{3}\ln(8) - \tfrac{32}{9}\right\} - \left\{\tfrac{2}{3}\ln(2) - \tfrac{4}{9}\right\}\right] \\
&= \tfrac{16}{3}\ln\left(2^3\right) - \tfrac{32}{9} - \tfrac{2}{3}\ln(2) + \tfrac{4}{9} \\
&= \tfrac{48}{3}\ln(2) - \tfrac{28}{9} - \tfrac{2}{3}\ln(2) \\
&= \tfrac{46}{3}\ln(2) - \tfrac{28}{9}
\end{aligned}
$$

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Question 2

The Question

(a) Use integration to find

$$\int \frac{1}{x^3} \ln(x)\ dx$$

(5 marks)

(b) Hence calculate

$$\int_{x=1}^{x=2} \frac{1}{x^3} \ln(x)\ dx$$

(2 marks)

My Answer

(a) Well, this is a classic integration by parts. Whenever we have a polynomial multiplied by a sine, cosine, exponential or log, it's parts. So...DIS is how we do it. As we don't know how to integrate $\ln(x)$, that has to go in the D column:

D	I	S
$\ln(x)$	x^{-3}	$+$
$\frac{1}{x}$	$-\frac{1}{2}x^{-2}$	$-$
		$+$

Figure 42: Integrating $\int 1/x^3 \ln(x)\ dx$

So our integral becomes

$$\begin{aligned}\int \frac{1}{x^3} \ln(x)\ dx &= -\frac{1}{2x^2}\ln(x) + \int \frac{1}{x} \cdot \frac{1}{2x^2}\ dx \\ &= -\frac{1}{2x^2}\ln(x) + \frac{1}{2}\int \frac{1}{x^3}\ dx \\ &= -\frac{1}{2x^2}\ln(x) + \frac{1}{2} \cdot -\frac{1}{2x^2}\ dx \\ &= -\frac{1}{2x^2}\ln(x) - \frac{1}{4x^2} + C\end{aligned}$$

(b) So this will be

$$\begin{aligned}\int_{x=1}^{x=2} \frac{1}{x^3} \ln(x)\ dx &= \left[-\frac{1}{2x^2}\ln(x) - \frac{1}{4x^2}\right]_{x=1}^{x=2} \\ &= \left[\left\{-\frac{1}{2\cdot 2^2}\ln(2) - \frac{1}{4\cdot 2^2}\right\} - \left\{-\frac{1}{2\cdot 1^2}\ln(1) - \frac{1}{4\cdot 1^2}\right\}\right] \\ &= \left[\left\{-\frac{1}{8}\ln(2) - \frac{1}{16}\right\} - \left\{-\frac{1}{2}\ln(1) - \frac{1}{4}\right\}\right] \\ &= -\tfrac{1}{8}\ln(2) - \tfrac{1}{16} + 0 + \tfrac{1}{4} \\ &= -\tfrac{1}{8}\ln(2) - \tfrac{1}{16} + \tfrac{4}{16} \\ &= -\tfrac{1}{8}\ln(2) + \tfrac{3}{16} \\ &= \tfrac{1}{16}\left[3 - 2\ln(2)\right]\end{aligned}$$

Question 4

The Question

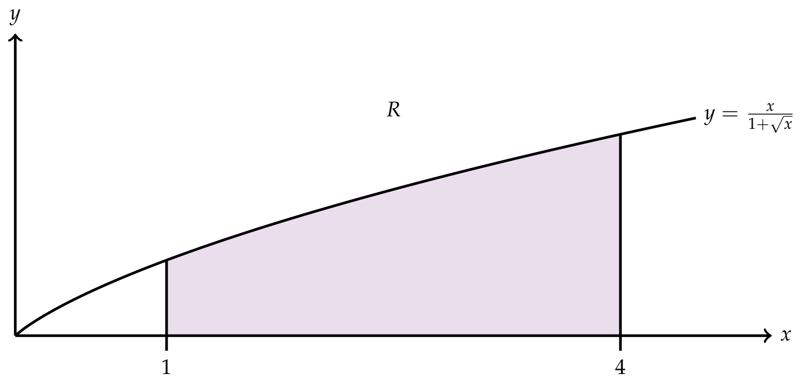

Figure 43: A graph of $y = \frac{x}{1+\sqrt{x}}$

Figure 43 shows a sketch of part of the curve with equation $y = \frac{x}{1+\sqrt{x}}$. The finite region R, shown shaded in Figure 43, is bounded by the curve, the x-axis, the line with equation $x = 1$ and the line with equation $x = 4$.

(a) ...

(b) ...

(c) Use the substitution $u = 1 + \sqrt{x}$, to find, by integrating, the exact area of R. **(8 marks)**

My Answer

(c) Well, they're telling us what to do. So let's do it...

If $\quad u = 1 + \sqrt{x} \quad$ then $\quad \frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} \quad$ and $\quad dx = 2x^{\frac{1}{2}}\ du$

So the integral we want will be

$$\int_{x=1}^{x=4} \frac{x}{1+\sqrt{x}}\ dx = \int_{x=1}^{x=4} \frac{x}{u} \cdot 2x^{\frac{1}{2}}\ du$$
$$= 2\int_{x=1}^{x=4} \frac{x^{\frac{3}{2}}}{u}\ du$$

We've still got x stuff in our integrand. What can we do with that? Well, the only thing we can do is to go back to the substitution and see if we can turn it into u stuff. Since $u = 1 + \sqrt{x}$, then $\sqrt{x} = u - 1$. That means that $(\sqrt{x})^3 = (u-1)^3$:

$$\int_{x=1}^{x=4} \frac{x}{1+\sqrt{x}}\ dx = 2\int_{x=1}^{x=4} \frac{(u-1)^3}{u}\ du$$

Now here, as we have a single u on the bottom, the easiest thing to do is to just multiply out the top and use the way fractions add/subtract:

$$\int_{x=1}^{x=4} \frac{x}{1+\sqrt{x}}\ dx = 2\int_{x=1}^{x=4} \frac{u^3 - 3u^2 + 3u - 1}{u}\ du$$
$$= 2\int_{x=1}^{x=4} u^2 - 3u + 3 - \frac{1}{u}\ du$$

and we now have nothing but standard forms to integrate:

$$\int_{x=1}^{x=4} \frac{x}{1+\sqrt{x}}\ dx = 2\left[\tfrac{1}{3}u^3 - \tfrac{3}{2}u^2 + 3u - \ln(u)\right]_{x=1}^{x=4}$$

Now we have to worry about the limits. When $x = 1$, $u = 2$, and when $x = 4$, $u = 3$, so

$$\begin{aligned}
\int_{x=1}^{x=4} \frac{x}{1+\sqrt{x}}\,dx &= 2\left[\tfrac{1}{3}u^3 - \tfrac{3}{2}u^2 + 3u - \ln(u)\right]_{u=2}^{u=3} \\
&= 2\left[\left\{\tfrac{1}{3}\cdot 3^3 - \tfrac{3}{2}\cdot 3^2 + 3\cdot 3 - \ln(3)\right\} - \left\{\tfrac{1}{3}\cdot 2^3 - \tfrac{3}{2}\cdot 2^2 + 3\cdot 2 - \ln(2)\right\}\right] \\
&= 2\left[\left\{9 - \tfrac{27}{2} + 9 - \ln(3)\right\} - \left\{\tfrac{8}{3} - 6 + 6 - \ln(2)\right\}\right] \\
&= 2\left[\left\{18 - \tfrac{27}{2} - \ln(3)\right\} - \left\{\tfrac{8}{3} - \ln(2)\right\}\right] \\
&= 2\left[\left\{\tfrac{36}{2} - \tfrac{27}{2} - \ln(3)\right\} - \left\{\tfrac{8}{3} - \ln(2)\right\}\right] \\
&= 2\left[\tfrac{9}{2} - \ln(3) - \tfrac{8}{3} + \ln(2)\right] \\
&= 2\left[\tfrac{27}{6} - \tfrac{16}{6} + \ln\left(\tfrac{2}{3}\right)\right] \\
&= 2\left[\tfrac{11}{6} + \ln\left(\tfrac{2}{3}\right)\right] \\
&= \tfrac{11}{3} + 2\ln\left(\tfrac{2}{3}\right)
\end{aligned}$$

Question 5

The Question

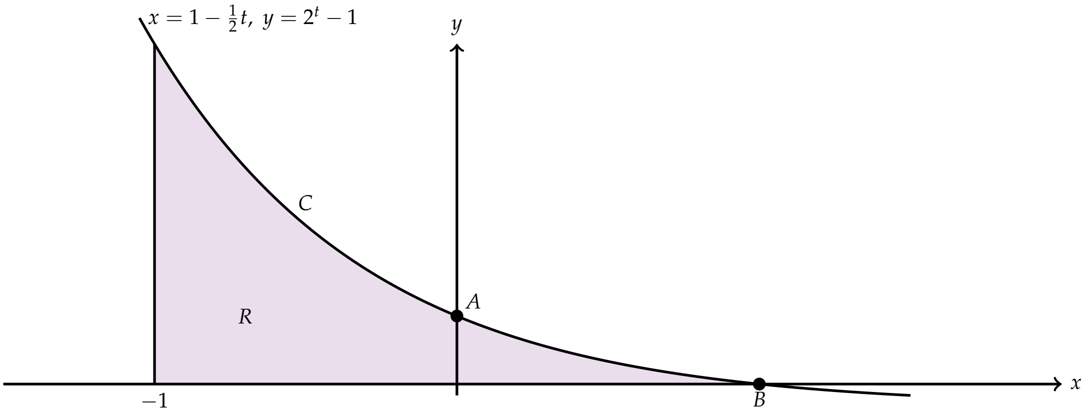

Figure 44: A graph of $x = 1 - \frac{1}{2}t,\ y = 2^t - 1$

Figure 44 shows a sketch of part of the curve C with parametric equations

$$x = 1 - \tfrac{1}{2}t, \qquad y = 2^t - 1$$

The curve crosses the y-axis at the point A and crosses the x-axis at the point B.

(a) ...

(b) Find the x-coordinate of the point B. **(2 marks)**

(c) ...

The region R, as shown shaded in Figure 44, is bounded by the curve C, the line $x = -1$ and the x-axis.

(d) Use integration to find the exact area of R. **(6 marks)**

My Answer

(b) B is at the place where $y = 0$, so

$$\begin{aligned}
2^t - 1 &= 0 \\
2^t &= 1 \\
t &= 0
\end{aligned}$$

And if $t = 0$ at B, then $x = 1$ at B.

(d) The integral we want will be

$$V = \int_{x=-1}^{x=1} y \, dx$$

And we have to get this in terms of t everywhere. The y bit is easy: that's just going to be $2^t - 1$. To get dx in terms of dt we differentiate $x = 1 - \frac{1}{2}t$:

$$\begin{aligned} x &= 1 - \tfrac{1}{2}t \\ \frac{dx}{dt} &= -\tfrac{1}{2} \\ dx &= -\tfrac{1}{2} \, dt \end{aligned}$$

Plugging this into the integral gives

$$\begin{aligned} V &= \int_{x=-1}^{x=1} \left(2^t - 1\right) \cdot -\tfrac{1}{2} \, dt \\ &= -\tfrac{1}{2} \int_{x=-1}^{x=1} 2^t - 1 \, dt \end{aligned}$$

The problem here is: how on earth do you integrate something like a^x??

This is the first time that the integral of a^x has come up in an exam for this specification. Because of that fact, you are just going to have to know this one. See a full explanation in Appendix B. You will have to consider it to be a standard form that you will have to learn.

$$V = -\tfrac{1}{2} \left[\frac{2^t}{\ln(2)} - t \right]_{x=-1}^{x=1}$$

Now for the limits. When $x = -1$, $t = 4$; when $x = 1$, $t = 0$ (as we found in part (b)). So...

$$\begin{aligned} V &= -\tfrac{1}{2} \left[\frac{2^t}{\ln(2)} - t \right]_{t=4}^{t=0} \\ &= \tfrac{1}{2} \left[\frac{2^t}{\ln(2)} - t \right]_{t=0}^{t=4} \end{aligned}$$

using that swapping the limits thing again. So:

$$\begin{aligned} V &= \tfrac{1}{2} \left[\left\{ \frac{2^4}{\ln(2)} - 4 \right\} - \left\{ \frac{2^0}{\ln(2)} - 0 \right\} \right] \\ &= \tfrac{1}{2} \left[\frac{16}{\ln(2)} - 4 - \frac{1}{\ln(2)} \right] \\ &= \tfrac{1}{2} \left[\frac{15}{\ln(2)} - 4 \right] \\ &= \frac{15}{2\ln(2)} - 2 \end{aligned}$$

Question 6

The Question

Figure 45 shows a sketch of part of the curve with equation $y = 1 - 2\cos(x)$, where x is measured in radians. The curve crosses the x-axis at the point A and at the point B.

(a) Find, in terms of π, the x-coordinate of the point A and the x-coordinate of the point B. **(3 marks)**

The finite region S enclosed by the curve and the x-axis is shown shaded in Figure 45. The region S is rotated through 2π radians about the x-axis.

(b) Find, by integration, the exact value of the volume of the solid generated. **(6 marks)**

My Answer

(a) A and B will be at palces where $y = 0$. So

$$\begin{aligned} 0 &= 1 - 2\cos(x) \\ 2\cos(x) &= 1 \\ \cos(x) &= \tfrac{1}{2} \end{aligned}$$

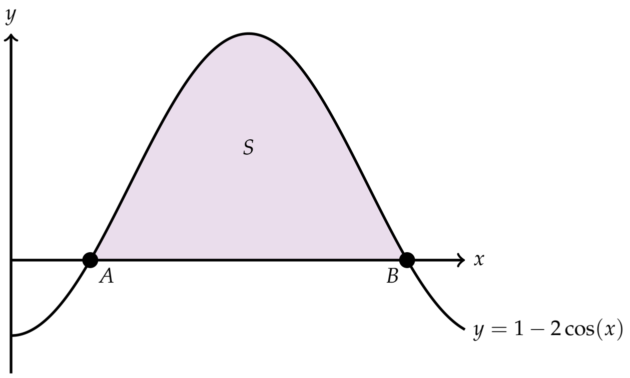

Figure 45: A graph of $y = 1 - 2\cos(x)$

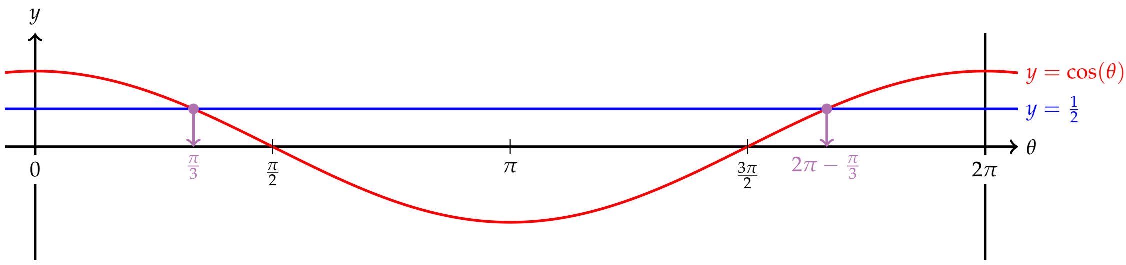

Figure 46: A graph of $y = \cos(\theta)$

Using my calculator, I find that $x = \frac{\pi}{3}$ will be one of these values. But what about the other one? Well, what I would do would be to draw the $y = \cos(\theta)$ graph:

Using the symmetry of the graph, the other solution will be $y = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$.

(b) The integral that we need to find will be

$$\begin{aligned} V &= \pi \int y^2 \, dx \\ &= \pi \int_{x=\frac{\pi}{3}}^{x=\frac{5\pi}{3}} \left[1 - 2\cos(x)\right]^2 \, dx \\ &= \pi \int_{x=\frac{\pi}{3}}^{x=\frac{5\pi}{3}} 1 - 4\cos(x) + 4\cos^2(x) \, dx \end{aligned}$$

Now the first two terms in this integrand are standard forms. But what about integrating $\cos^2(x)$? Well, when you have $\sin^2(x)$ or $\cos^2(x)$ to integrate, there is a trick. It's not easy to integrate powers of $\sin(x)$ and $\cos(x)$ higher than one, so we have to convert them to single powers using trigonometrical identities. And the identities are: $\mathbf{\cos(2x) \equiv 2\cos^2(x) - 1 \equiv 1 - 2\sin^2(x)}$ (from the addition formulae). Using these identities we can convert either $\sin^2(x)$ or $\cos^2(x)$ into something with a $\cos(2x)$ in it.

So how does that idea work here? Let's see. If $\cos(2x) \equiv 2\cos^2(x) - 1$, then $\cos(2x) + 1 \equiv 2\cos^2(x)$ and so $2\cos(2x) + 2 \equiv 4\cos^2(x)$. So our integral will become

$$V = \pi \int_{x=\frac{\pi}{3}}^{x=\frac{5\pi}{3}} 1 - 4\cos(x) + 2\cos(2x) + 2 \, dx$$

And everything in the integral is now a standard form. Well, almost. I'm hoping by now you can integrate $\cos(2x)$ without needeing to resort to substitution. So,

$$\begin{aligned} V &= \pi \int_{x=\frac{\pi}{3}}^{x=\frac{5\pi}{3}} 3 - 4\cos(x) + 2\cos(2x) \, dx \\ &= \pi \Big[3x - 4\sin(x) + \sin(2x)\Big]_{x=\frac{\pi}{3}}^{x=\frac{5\pi}{3}} \end{aligned}$$

Now we can plug the numbers in

$$\begin{aligned} V &= \pi\Big[\Big\{3\cdot\tfrac{5\pi}{3} - 4\sin\left(\tfrac{5\pi}{3}\right) + \sin\left(2\cdot\tfrac{5\pi}{3}\right)\Big\} - \Big\{3\cdot\tfrac{\pi}{3} - 4\sin\left(\tfrac{\pi}{3}\right) + \sin\left(2\cdot\tfrac{\pi}{3}\right)\Big\}\Big] \\ &= \pi\Big[\Big\{5\pi + 2\sqrt{3} - \tfrac{\sqrt{3}}{2}\Big\} - \Big\{\pi - 2\sqrt{3} + \tfrac{\sqrt{3}}{2}\Big\}\Big] \\ &= \pi\Big[5\pi + 2\sqrt{3} - \tfrac{\sqrt{3}}{2} - \pi + 2\sqrt{3} - \tfrac{\sqrt{3}}{2}\Big] \\ &= \pi\Big[4\pi + 3\sqrt{3}\Big] \end{aligned}$$

Tips, Tricks and Patterns

Tricks:

When you have $\sin^2(x)$ or $\cos^2(x)$ to integrate, there is a trick.

You have to use the identities: $\mathbf{\cos(2x) \equiv 2\cos^2(x) - 1 \equiv 1 - 2\sin^2(x)}$.

Question 8

The Question

A bottle of water is put into a refrigerator. The temperature inside the refrigerator remains constant at 3°C and t minutes after the bottle is placed in the refrigerator the temperature of the water in the bottle is θ°C.

The rate of change of the temperature of the water in the bottle is modelled by the differential equation

$$\frac{d\theta}{dt} = \frac{3-\theta}{125}$$

(a) By solving the differential equation, show that

$$\theta = Ae^{-0.008t} + 3$$

where A is a constant. **(4 marks)**

Given that the temperature of the water in the bottle when it was put in the refrigerator was 16°C,

(b) find the time taken for the temperature of the water in the bottle to fall to 10°C, giving your answer to the nearest minute. **(5 marks)**

My Answer

(a) The only way we know how to solve differential equations at A-level is to separate the variables

$$\frac{1}{3-\theta}\,d\theta = \frac{1}{125}\,dt$$

and integrate both sides

$$\int \frac{1}{3-\theta}\,d\theta = \int \frac{1}{125}\,dt$$

Now on the left I've spotted that we can use that

$$\int \frac{f'(x)}{f(x)}\,dx = \ln\left[f(x)\right]$$

trick. For that to work th top of the integrand needs to be -1. Easily fixed:

$$-\int \frac{-1}{3-\theta}\,d\theta = \frac{1}{125}\int 1\,dt$$

and so

$$\int \frac{-1}{3-\theta}\,d\theta = -\frac{1}{125}\int 1\,dt$$

Why have I switched the $-$ sign to the right hand side? Because of the answer... Right, off we go then:

$$\ln(3-\theta) = -\frac{1}{125}t + \ln(C)$$

Why $\ln(C)$ rather than C? Experience! See what happens...

$$\begin{aligned}\ln(3-\theta)-\ln(C) &= -\frac{1}{125}t\\ \ln\left(\frac{3-\theta}{C}\right) &= -\frac{1}{125}t\\ \frac{3-\theta}{C} &= e^{-\frac{1}{125}t}\\ 3-\theta &= Ce^{-\frac{1}{125}t}\\ \theta &= 3-Ce^{-\frac{1}{125}t}\end{aligned}$$

which is of the right form. I could set my C to be equal to $-A$, and I'm home. Notice that $\frac{1}{125}=0.008$.

(b) For this part we need to find a time, so we need to make t the subject of this equation. Instead of starting with the result from part (a), let's back track a bit: in our processing we came up with

$$\ln\left(\frac{3-\theta}{C}\right) = -\frac{1}{125}t$$

which I think is an easier place to start! So...

$$\begin{aligned}t &= -125\ln\left(\frac{3-\theta}{C}\right)\\ &= 125\ln\left(\frac{C}{3-\theta}\right)\end{aligned}$$

using the log rules. Now we are told that when $t=0$, $\theta=16^\circ$, so

$$\begin{aligned}0 &= 125\ln\left(\frac{C}{3-16}\right)\\ 0 &= \ln\left(\frac{C}{-13}\right)\end{aligned}$$

so $C=-13$, since the right hand side must be $\ln(1)$. So

$$\begin{aligned}t &= 125\ln\left(\frac{-13}{3-\theta}\right)\\ &= 125\ln\left(\frac{13}{\theta-3}\right)\end{aligned}$$

So now we need the time when $\theta=10^\circ$:

$$\begin{aligned}t &= 125\ln\left(\frac{13}{10-3}\right)\\ &= 77 \text{ minutes, to the nearest minute.}\end{aligned}$$

Tips, Tricks and Patterns

Patterns:

$$\int f'(x)\cdot[f(x)]^n\,dx,\qquad \int f'(x)\cdot e^{f(x)}\,dx,\qquad \int \frac{f'(x)}{[f(x)]^n}\,dx \qquad \text{indeed...} \qquad \int f'(x)\cdot g[f(x)]\,dx$$

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Question 1

The Question

(a) Find

$$\int x^2 e^x \, dx$$

(5 marks)

(b) Hence find the exact value of

$$\int_{x=0}^{x=1} x^2 e^x \, dx$$

(2 marks)

My Answer

(a) Well, this is a classic integration by parts. Whenever we have a polynomial multiplied by a sine, cosine, exponential or log, it's parts. So...DIS is how we do it:

D	I	S
x^2	e^x	$+$
$2x$	e^x	$-$
2	e^x	$+$
0	e^x	$-$
		$+$

Figure 47: Integrating $\int x^2 e^x \, dx$

So our integral becomes

$$\begin{aligned}\int x^2 e^x \, dx &= x^2 e^x - 2xe^x + 2e^x + C \\ &= e^x \Big[x^2 - 2x + 2\Big] + C\end{aligned}$$

(b) For this part we are evaluating

$$\begin{aligned}\int_{x=0}^{x=1} x^2 e^x \, dx &= \Big[e^x \left[x^2 - 2x + 2\right]\Big]_{x=0}^{x=1} \\ &= \Big[\Big\{e^1 \left[1^2 - 2 \cdot 1 + 2\right]\Big\} - \Big\{e^0 \left[0^2 - 2 \cdot 0 + 2\right]\Big\}\Big] \\ &= \Big[\Big\{e\Big\} - \Big\{2\Big\}\Big] \\ &= e - 2\end{aligned}$$

Question 3

The Question

Figure 48 shows the finite region R bounded by the x-axis, the y-axis, the line $x = \frac{\pi}{2}$ and the curve with equation

$$y = \sec\left(\tfrac{1}{2}x\right), \qquad 0 \le x \le \tfrac{\pi}{2}$$

(a) ...

(b) ...

Region R is rotated through 2π radians about the x-axis.

(c) Use calculus to find the exact volume of the solid formed. **(4 marks)**

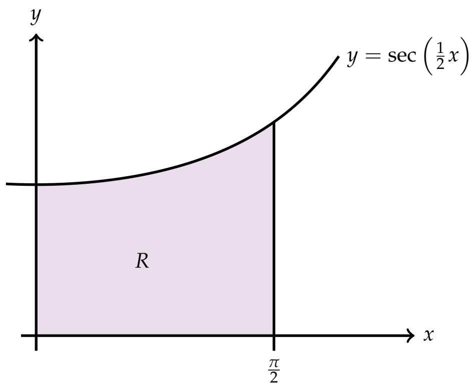

Figure 48: A graph of $y = \sec\left(\frac{1}{2}x\right)$

My Answer

(c) The integral we want will be

$$\begin{aligned} V &= \pi \int y^2 \, dx \\ &= \pi \int_{x=0}^{x=\frac{\pi}{2}} \sec^2\left(\frac{1}{2}x\right) \, dx \end{aligned}$$

Well, $\sec^2(x)$ is a standard form, $\sec^2\left(\frac{1}{2}x\right)$ isn't. But it's very easy to transform it into a standard form:

$$\text{If} \qquad u = \tfrac{1}{2}x \qquad \text{then} \qquad \frac{du}{dx} = \frac{1}{2} \qquad \text{and} \qquad dx = 2\,du$$

So our integral becomes

$$\begin{aligned} V &= \pi \int_{x=0}^{x=\frac{\pi}{2}} \sec^2(u) \cdot 2 \, du \\ &= 2\pi \int_{x=0}^{x=\frac{\pi}{2}} \sec^2(u) \, du \end{aligned}$$

We can now integrate this

$$\begin{aligned} V &= 2\pi \Big[\tan(u) \Big]_{x=0}^{x=\frac{\pi}{2}} \\ &= 2\pi \left[\tan\left(\tfrac{1}{2}x\right) \right]_{x=0}^{x=\frac{\pi}{2}} \end{aligned}$$

And putting the numbers in

$$\begin{aligned} V &= 2\pi \left[\left\{ \tan\left(\tfrac{1}{2} \cdot \tfrac{\pi}{2}\right) \right\} - \left\{ \tan\left(\tfrac{1}{2} \cdot 0\right) \right\} \right] \\ &= 2\pi \left[\left\{ \tan\left(\tfrac{\pi}{4}\right) \right\} - \left\{ \tan(0) \right\} \right] \\ &= 2\pi \left[1 - 0 \right] \\ &= 2\pi \end{aligned}$$

Question 5

The Question

(a) Use the substitution $x = u^2$, $u > 0$, to show that

$$\int \frac{1}{x(2\sqrt{x} - 1)} \, dx = \int \frac{2}{u(2u - 1)} \, du$$

(3 marks)

(b) Hence show that

$$\int_{x=1}^{x=9} \frac{1}{x(2\sqrt{x}-1)}\, dx = 2\ln\left(\frac{a}{b}\right)$$

where a and b are integers to be determined. **(7 marks)**

My Answer

(a) So again, they're telling that this is a substitution, and what substitution to pick. So...

If $x = u^2$ then $\frac{dx}{du} = 2u$ and $dx = 2u\, du$

Shoving this into the integral we get

$$\begin{aligned}\int \frac{1}{x(2\sqrt{x}-1)}\, dx &= \int \frac{1}{u^2(2u-1)} \cdot 2u\, du \\ &= \int \frac{2}{u(2u-1)}\, du\end{aligned}$$

using the fact that if $x = u^2$, then $\sqrt{x} = u$.

(b) So now we've got to integrate this thing, and they give no clue as to which method to use. It's probably not going to be a trigonometrical substitution. It's probably not going to be parts. That leaves substitution and partial fractions. Hang on, partial fractions! Yes, this looks like one of those pesky partial fraction things.

That means we have to do this: find the A and the B that will make this work:

$$\frac{2}{u(2u-1)} = \frac{A}{u} + \frac{B}{2u-1}$$

OK, add the two fractions together by finding equivalent fractions with the required common denominator:

$$\begin{aligned}\frac{2}{u(2u-1)} &= \frac{A(2u-1)}{u(2u-1)} + \frac{Bu}{u(2u-1)} \\ &= \frac{A(2u-1)+Bu}{u(2u-1)}\end{aligned}$$

Then we just need to compare the tops:

$$\begin{aligned}2 &= A(2u-1) + Bu \\ 2 &= (2A+B)u - A\end{aligned}$$

so that $A = -2$, and $2A + B = 0$. That would mean that $B = 4$. So

$$\frac{2}{u(2u-1)} = -\frac{2}{u} + \frac{4}{2u-1}$$

That means that we can write our integral like this

$$\begin{aligned}\int \frac{2}{u(2u-1)}\, du &= \int -\frac{2}{u} + \frac{4}{2u-1}\, du \\ &= -2\int \frac{1}{u}\, du + 2\int \frac{2}{2u-1}\, du\end{aligned}$$

Why have I written the integrals like this? This is why:

$$\int \frac{f'(x)}{f(x)}\, dx = \ln\left[f(x)\right]$$

remember? So...

$$\int \frac{2}{u(2u-1)}\, du = -2\ln(u) + 2\ln(2u-1) + C$$

So now we need to put the limits in. When $x = 1$, $u = 1$, and when $x = 9$, $u = 3$, so...

$$\begin{aligned}\int_{x=1}^{x=9} \frac{2}{u(2u-1)}\, du &= \Big[-2\ln(u) + 2\ln(2u-1)\Big]_{u=1}^{u=3} \\ &= \Big[\Big\{-2\ln(3) + 2\ln(2\cdot 3-1)\Big\} - \Big\{-2\ln(1) + 2\ln(2\cdot 1-1)\Big\}\Big] \\ &= \Big[\Big\{-2\ln(3) + 2\ln(5)\Big\} - \Big\{0 + 2\ln(1)\Big\}\Big] \\ &= 2\ln\left(\tfrac{5}{3}\right)\end{aligned}$$

Tips, Tricks and Patterns

Patterns:

$$\int f'(x) \cdot [f(x)]^n \, dx, \qquad \int f'(x) \cdot e^{f(x)} \, dx, \qquad \int \frac{f'(x)}{[f(x)]^n} \, dx \qquad \text{indeed...} \qquad \int f'(x) \cdot g[f(x)] \, dx$$

Question 6

The Question

Water is being heated in a kettle. At time t seconds, the temperature of the water is θ°C. The rate of increase of the temperature of the water at any time t is modelled by the differential equation

$$\frac{d\theta}{dt} = \lambda(120 - \theta), \qquad \theta \leq 100$$

where λ is a positive constant.

Given that $\theta = 20$ when $t = 0$,

(a) solve this differential equation to show that

$$\theta = 120 - 100e^{-\lambda t}$$

(8 marks)

When the temperature of the water reaches 100°C, the kettle switches off.

(b) Given that $\lambda = 0.01$, find the time, to the nearest second, when the kettle switches off. **(3 marks)**

My Answer

(a) The only way we know how to solve differential equations at A-Level is to separate the variables

$$\frac{1}{120 - \theta} \, d\theta = \lambda \, dt$$

and integrate both sides

$$\int \frac{1}{120 - \theta} \, d\theta = \int \lambda \, dt$$

I'm going to use that old

$$\int \frac{f'(x)}{f(x)} \, dx = \ln [f(x)]$$

pattern again on the left hand side, so

$$\int \frac{-1}{120 - \theta} \, d\theta = -\int \lambda \, dt$$

this time multiplying both sides by -1. Then we can integrate

$$\ln(120 - \theta) = -\lambda t + \ln(C)$$

again using $\ln(C)$ as my constant when there is a natural log involved. So...

$$\begin{aligned}
\ln(120 - \theta) - \ln(C) &= -\lambda t \\
\ln\left(\frac{120 - \theta}{C}\right) &= -\lambda t \\
\frac{120 - \theta}{C} &= e^{-\lambda t} \\
120 - \theta &= Ce^{-\lambda t} \\
\theta &= 120 - Ce^{-\lambda t}
\end{aligned}$$

To find the C we know that $\theta = 20$ when $t = 0$, so

$$\begin{aligned}
20 &= 120 - Ce^{0} \\
20 &= 120 - C \\
C &= 100
\end{aligned}$$

So our solution is

$$\theta = 120 - 100e^{-\lambda t}$$

(b) For this bit we need to make t the subject. Instead of going from the general solution, I'm going to start from a bit further back in the processing...namely to this bit

$$\ln(120 - \theta) - \ln(C) = -\lambda t$$

Putting in the values of λ and C we get

$$\ln(120 - \theta) - \ln(100) = -\frac{1}{100}t$$

and so

$$\begin{aligned} t &= -100\Big[\ln(120 - \theta) - \ln(100)\Big] \\ &= -100 \ln\left(\frac{120 - \theta}{100}\right) \end{aligned}$$

so when $\theta = 100$,

$$\begin{aligned} t &= -100 \ln\left(\frac{120 - 100}{100}\right) \\ &= 161 \text{ seconds, to the nearest second.} \end{aligned}$$

Tips, Tricks and Patterns

Patterns:

$$\int f'(x) \cdot [f(x)]^n \, dx, \qquad \int f'(x) \cdot e^{f(x)} \, dx, \qquad \int \frac{f'(x)}{[f(x)]^n} \, dx \qquad \text{indeed...} \qquad \int f'(x) \cdot g[f(x)] \, dx$$

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18 Edexcel C4 June 2013 (Replacement)

Question 3

The Question

Using the substitution $u = 2 + \sqrt{2x+1}$, or other suitable substitutions, find the exact value of

$$I = \int_{x=0}^{x=4} \frac{1}{2+\sqrt{2x+1}}\, dx$$

giving your answer in the form $A + 2\ln(B)$, where A is an integer and B is a positive constant. **(8 marks)**

My Answer

(a) Well, it's probably substitution, then! And guess what? I reckon (just guessing, you know) that the substitution will be

If $u = 2 + \sqrt{2x+1}$ then $\frac{du}{dx} = \frac{1}{2}(2x+1)^{-\frac{1}{2}} \times 2$ and $dx = (2x+1)^{\frac{1}{2}}\, du$

Right, let's plug what we know in:

$$I = \int_{x=0}^{x=4} \frac{1}{u} \cdot (2x+1)^{\frac{1}{2}}\, du$$

But from the substitution, $(2x+1)^{\frac{1}{2}} = u - 2$, so

$$I = \int_{x=0}^{x=4} \frac{u-2}{u}\, du$$

This is another of those where we can just use the way fractions add/subtract:

$$I = \int_{x=0}^{x=4} 1\, du - 2\int_{x=0}^{x=4} \frac{1}{u}\, du$$

And now we only have standard forms in the integrals, so we can integrate:

$$\begin{aligned} I &= \Big[u\Big]_{x=0}^{x=4} - 2\Big[\ln(u)\Big]_{x=0}^{x=4} \\ &= \Big[u - 2\ln(u)\Big]_{x=0}^{x=4} \end{aligned}$$

Plugging the numbers in, we get

$$I = \Big[\Big\{u - 2\ln(u)\Big\} - \Big\{u - 2\ln(u)\Big\}\Big]_{x=0}^{x=4}$$

Now we have to worry about the limits. Going back to the substitution, when $x = 0$, $u = 3$, and when $x = 4$, $u = 5$, and so

$$\begin{aligned} I &= \Big[\Big\{u - 2\ln(u)\Big\} - \Big\{u - 2\ln(u)\Big\}\Big]_{u=3}^{u=5} \\ &= \Big[\Big\{5 - 2\ln(5)\Big\} - \Big\{3 - 2\ln(3)\Big\}\Big]_{u=3}^{u=5} \\ &= 5 - 2\ln(5) - 3 + 2\ln(3) \\ &= 2 + 2\ln\left(\frac{3}{5}\right) \end{aligned}$$

Question 5

The Question

Figure 49 shows part of the curve with the equation $y = 4te^{-\frac{1}{3}t} + 3$. The finite region R shown shaded in Figure 49 is bounded by the curve, the x-axis, the t-axis and the line $t = 8$.

(a) ...

(b) ...

(c) Use calculus to find the exact value for the area of R. **(6 marks)**

(d) ...

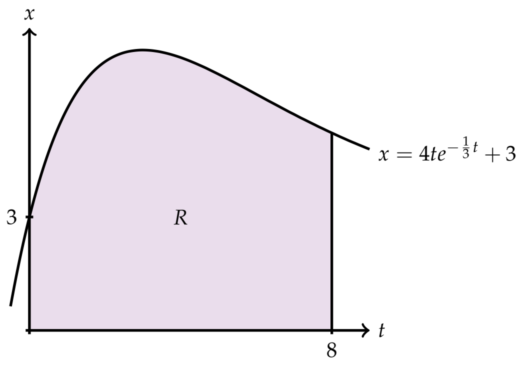

Figure 49: A graph of $x = 4te^{-\frac{1}{3}t} + 3$

My Answer

(c) The integral we will need is

$$R = \int_{t=0}^{t=8} 4te^{-\frac{1}{3}t} + 3 \, dt$$
$$= \int_{t=0}^{t=8} 4te^{-\frac{1}{3}t} \, dt + \int_{t=0}^{t=8} 3 \, dt$$

The second integral is a standard form. The first integral is a classic integration by parts. Whenever we have a polynomial multiplied by a sine, cosine, exponential or log, it's parts. So...check out Figure 50 for DIS is how we do it!

D	I	S
$4t$	$e^{-\frac{1}{3}t}$	$+$
4	$-3e^{-\frac{1}{3}t}$	$-$
0	$9e^{-\frac{1}{3}t}$	$+$
		$-$

Figure 50: Integrating $\int 4te^{-\frac{1}{3}t} \, dt$

So our integral becomes

$$R = \Big[-12te^{-\frac{1}{3}t} - 36e^{-\frac{1}{3}t} \Big]_{t=0}^{t=8} + \Big[3t \Big]_{t=0}^{t=8}$$
$$= \Big[-12te^{-\frac{1}{3}t} - 36e^{-\frac{1}{3}t} + 3t \Big]_{t=0}^{t=8}$$

Now we can plug the numbers in:

$$R = \Big[\Big\{ -12 \cdot 8 \cdot e^{-\frac{1}{3} \cdot 8} - 36e^{-\frac{1}{3} \cdot 8} + 3 \cdot 8 \Big\} - \Big\{ -12 \cdot 0 \cdot e^{-\frac{1}{3} \cdot 0} - 36e^{-\frac{1}{3} \cdot 0} + 3 \cdot 0 \Big\} \Big]$$
$$= \Big[\Big\{ -96 \cdot e^{-\frac{8}{3}} - 36e^{-\frac{8}{3}} + 24 \Big\} - \Big\{ 0 - 36 + 0 \Big\} \Big]$$
$$= 60 - 132e^{-\frac{8}{3}}$$

Question 7

The Question

Figure 51 shows a sketch of a curve C with parametric equations

$$x = 27\sec^3(t), \qquad y = \tan(t), \qquad 0 \le t \le \frac{\pi}{3}$$

(a) ...

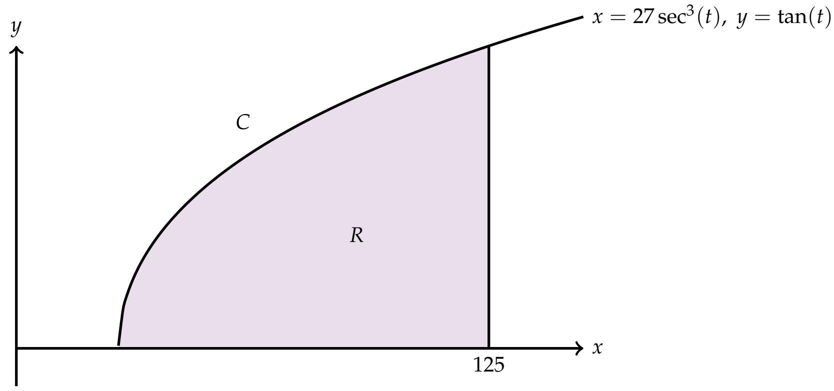

Figure 51: A graph of $x = 27\sec^3(t),\ y = \tan(t)$

(b) Show that the cartesian equation of C may be written in the form

$$y = \sqrt{x^{\frac{2}{3}} - 9}, \qquad a \le x \le b$$

stating the values of a and b. **(3 marks)**

The finite region R which is bounded by the curve C, the x-axis and the line $x = 125$ is shown shaded in Figure 51. This region is rotated through 2π radians about the x-axis to form a solid of revolution.

(c) Use calculus to find the exact value of the volume of the solid of revolution. **(5 marks)**

My Answer

(b) When $y = 0$, then $t = 0$, and so $x = 27\sec^3(0) = 27$. That gives us the lower bound for x.

(c) The integral we want will be

$$\begin{aligned} V &= \pi \int y^2 \, dx \\ &= \pi \int_{x=27}^{x=125} x^{\frac{2}{3}} - 9 \, dx \end{aligned}$$

Well, this is just standard form stuff, so we can integrate it:

$$V = \pi \left[\tfrac{3}{5} x^{\frac{5}{3}} - 9x \right]_{x=27}^{x=125}$$

Now let's plug the numbers in:

$$\begin{aligned} V &= \pi \left[\left\{ \tfrac{3}{5} \cdot 125^{\frac{5}{3}} - 9 \cdot 125 \right\} - \left[\left\{ \tfrac{3}{5} \cdot 27^{\frac{5}{3}} - 9 \cdot 27 \right\} \right] \right. \\ &= \pi \left[\left\{ \tfrac{3}{5} \cdot 3125 - 1125 \right\} - \left[\left\{ \tfrac{3}{5} \cdot 243 - 243 \right\} \right] \right. \\ &= \tfrac{4236}{5} \pi \end{aligned}$$

Question 8

The Question

In an experiment testing solid rocket fuel, some fuel is burned and the waste products are collected. Throughout the experiment the sum of the masses of the unburned fuel and waste products remains constant.

Let x be the mass of waste products, in kg, at time t minutes after the start of the experiment. It is known that at time t minutes, the rate of increase of the mass of waste products, in kg per minute, is k times the mass of the unburned fuel remaining, where k is a positive constant.

The differential equation connecting x and t may be written in the form

$$\frac{dx}{dt} = k(M - x), \qquad \text{where } M \text{ is a constant.}$$

(a) ...

(b) Initially, the mass of waste products is zero. Solve the differential equation, expressing x in terms of k, M and t. **(6 marks)**

(c) If $x = \frac{1}{2}M$ when $t = \ln(4)$, find the value of x when $t = \ln(9)$, expressing x in terms of M, in its simplest form. **(4 marks)**

My Answer

(b) The only way we know how to solve differential equations at A-Level is to separate the variables

$$\frac{1}{M-x}\,dx = k\,dt$$

and integrate both sides

$$\int \frac{1}{M-x}\,dx = \int k\,dt$$

The right hand side is a standard form; the left hand side I'm spotting is an example of the old $\int \frac{f'(x)}{f(x)}\,dx = \ln\left[f(x)\right]$ so I'm going to multiply both sides by -1:

$$\int \frac{-1}{M-x}\,dx = -k\int 1\,dt$$

Now I can integrate:

$$\ln(M-x) = -kt + \ln(C)$$

Why $\ln(C)$?? My experience tells me do do this whenwever I have a $\ln(x)$ on the other side. So...

$$\begin{aligned} \ln(M-x) - \ln(C) &= -kt \\ \ln\left(\frac{M-x}{C}\right) &= -kt \\ \frac{M-x}{C} &= e^{-kt} \\ M-x &= Ce^{-kt} \\ x &= M - Ce^{-kt} \end{aligned}$$

To find the C we use the fact that when $t=0$, $x=0$ (the mass of the waste products is zero initially). So:

$$\begin{aligned} 0 &= M - Ce^{0} \\ C &= M \end{aligned}$$

So our solution is

$$\begin{aligned} x &= M - Me^{-kt} \\ &= M\left[1-e^{-kt}\right] \end{aligned}$$

(a) Now if $x = \frac{1}{2}M$ when $t = \ln(4)$, then we will be able to find the value of k:

$$\begin{aligned} \tfrac{1}{2}M &= M\left[1-e^{-k\cdot\ln(4)}\right] \\ \tfrac{1}{2} &= 1 - \left\{e^{\ln(4)}\right\}^{-k} \\ \tfrac{1}{2} &= 1 - 4^{-k} \\ 4^{-k} &= \tfrac{1}{2} \\ 2^{-2k} &= 2^{-1} \\ k &= \tfrac{1}{2} \end{aligned}$$

So, when $t = \ln(9)$,

$$\begin{aligned} x &= M\left[1-e^{-\frac{1}{2}\cdot\ln(9)}\right] \\ &= M\left[1-\left\{e^{\ln(9)}\right\}^{-\frac{1}{2}}\right] \\ &= M\left[1-9^{-\frac{1}{2}}\right] \\ &= M\left[1-\tfrac{1}{3}\right] \\ &= \tfrac{2}{3}M \end{aligned}$$

Tips, Tricks and Patterns

Patterns:

$$\int f'(x)\cdot\left[f(x)\right]^n\,dx, \qquad \int f'(x)\cdot e^{f(x)}\,dx, \qquad \int \frac{f'(x)}{\left[f(x)\right]^n}\,dx \qquad \text{indeed...} \qquad \int f'(x)\cdot g\left[f(x)\right]\,dx$$

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19 Edexcel C4 June 2014

Question 3

The Question

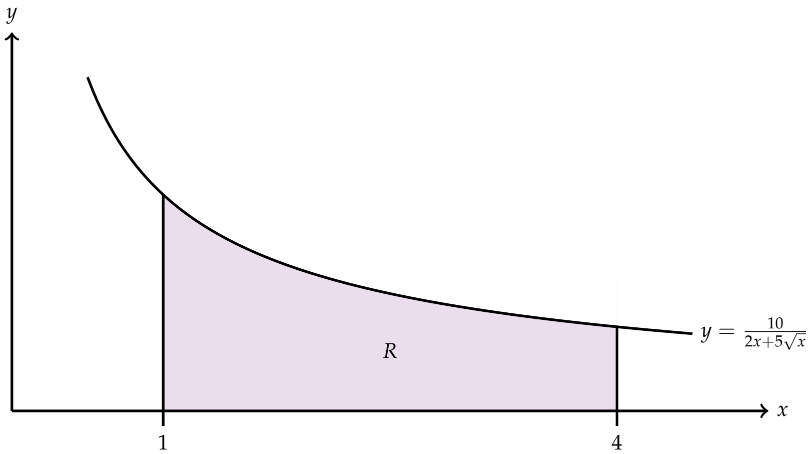

Figure 52: A graph of $y = \frac{10}{2x+5\sqrt{x}}$

Figure 52 shows a sketch of part of the curve with equation $y = \frac{10}{2x+5\sqrt{x}}, x > 0$. The finite region R, shown shaded in Figure 52, is bounded by the curve, the x-axis, and the lines with equations $x = 1$ and $x = 4$.

(a) ...

(b) ...

(c) ...

(d) Use the substitution $u = \sqrt{x}$, or otherwise, to find the exact value of

$$I = \int_{x=1}^{x=4} \frac{10}{2x + 5\sqrt{x}}\, dx$$

(6 marks)

My Answer

(d) Substitution again, then.

$$\text{If} \qquad u = \sqrt{x} = x^{\frac{1}{2}} \qquad \text{then} \qquad \frac{du}{dx} = \tfrac{1}{2}x^{-\frac{1}{2}} \qquad \text{and} \qquad dx = 2\sqrt{x}\, du$$

Plug it all in:

$$I = \int_{x=1}^{x=4} \frac{10}{2x + 5\sqrt{x}}\, dx = \int_{x=1}^{x=4} \frac{10}{2x + 5u} \cdot 2\sqrt{x} du$$
$$= \int_{x=1}^{x=4} \frac{20u}{2u^2 + 5u}\, du$$

using the fact that if $u = \sqrt{x}$, then $x = u^2$. Simplifying slightly,

$$I = \int_{x=1}^{x=4} \frac{20}{2u + 5}\, du$$

Ah - now I've spotted something again here...it's one of these:

$$\int \frac{f'(x)}{f(x)}\, dx = \ln\left[f(x)\right]$$

so I'll write my integral as

$$I = 10 \int_{x=1}^{x=4} \frac{2}{2u+5}\, du$$

because then I can just write the answer down:

$$\begin{aligned} I &= 10\Big[\ln(2u+5)\Big]_{x=1}^{x=4} \\ &= 10\Big[\ln(2\sqrt{x}+5)\Big]_{x=1}^{x=4} \end{aligned}$$

So now I just have the plug the numbers in:

$$\begin{aligned} I &= 10\Big[\Big\{\ln(2\sqrt{4}+5)\Big\} - \Big\{\ln(2\sqrt{1}+5)\Big\}\Big] \\ &= 10\Big[\Big\{\ln(9)\Big\} - \Big\{\ln(7)\Big\}\Big] \\ &= 10\ln\left(\tfrac{9}{7}\right) \end{aligned}$$

Tips, Tricks and Patterns

Patterns:

$$\int f'(x)\cdot[f(x)]^n\, dx, \qquad \int f'(x)\cdot e^{f(x)}\, dx, \qquad \int \frac{f'(x)}{[f(x)]^n}\, dx \qquad \text{indeed...} \qquad \int f'(x)\cdot g[f(x)]\, dx$$

Question 6

The Question

(i) Find

$$\int xe^{4x}\, dx$$

(3 marks)

(ii) Find

$$\int \frac{8}{(2x-1)^3}\, dx, \qquad x > \tfrac{1}{2}$$

(2 marks)

(iii) Given that $y = \frac{\pi}{6}$ at $x = 0$, solve the differential equation

$$\frac{dy}{dx} = e^x \operatorname{cosec}(2y) \operatorname{cosec}(y)$$

(7 marks)

My Answer

(i) Well, this is a classic integration by parts. Whenever we have a polynomial multiplied by a sine, cosine, exponential or log, it's parts. So...check out Figure 53 for DIS is how we do it!

D	I	S
x	e^{4x}	$+$
1	$\frac{1}{4}e^{4x}$	$-$
0	$\frac{1}{16}e^{4x}$	$+$
		$-$

Figure 53: Integrating $\int xe^{4x}\, dx$

So our integral becomes

$$\int xe^{4x}\,dx = \tfrac{1}{4}xe^{4x} - \tfrac{1}{16}e^{4x} + C$$

(ii) This looks a bit like x^{-3} to me. That gives me an idea...

$$\text{If} \qquad u = 2x-1 \qquad \text{then} \qquad \frac{du}{dx} = 2 \qquad \text{and} \qquad dx = \tfrac{1}{2}\,du$$

Plugging this in we get

$$\begin{aligned} I = \int \frac{8}{(2x-1)^3}\,dx &= \int \frac{8}{u^3}\cdot\frac{1}{2}\,du \\ &= \int \frac{4}{u^3}\,du \\ &= 4\int u^{-3}\,du \end{aligned}$$

which is a standard form. Integrating...

$$\begin{aligned} I &= 4\cdot -\tfrac{1}{2}u^{-2} + C \\ &= -\frac{2}{u^2} + C \\ &= -\frac{2}{(2x-1)^2} + C \end{aligned}$$

(iii) The only way we know how to solve differential equations at A-Level is to separate the variables

$$\frac{1}{\operatorname{cosec}(2y)\operatorname{cosec}(y)}\,dy = e^x\,dx$$

and integrate both sides

$$\int \frac{1}{\operatorname{cosec}(2y)\operatorname{cosec}(y)}\,dy = \int e^x\,dx$$

Now the right hand side is a standard form. But what about the left hand side? Well, using the definition of $\operatorname{cosec}(x)$ the left hand side becomes

$$\int \sin(2y)\sin(y)\,dy = \int e^x\,dx$$

Does that help? Now we've got a double angle here: can we get these pesky sin things to only have single y arguments? Yes:

$$\int 2\sin(y)\cos(y)\sin(y)\,dy = \int e^x\,dx$$

since $\sin(2y) = 2\sin(y)\cos(y)$. Right. Let's tidy things up a bit more:

$$2\int \sin^2(y)\cos(y)\,dy = \int e^x\,dx$$

And lookee here! We've got one of these things:

$$\int f'(x)\cdot[f(x)]^n\,dx \qquad \Rightarrow \qquad \text{substitution with } u = f(x)$$

So, let's use the substitution

$$\text{If} \qquad u = \sin(x) \qquad \text{then} \qquad \frac{du}{dx} = \cos x \qquad \text{and} \qquad dx = \tfrac{1}{\cos(x)}\,du$$

Plugging this in:

$$2\int u^2\cos(y)\cdot\tfrac{1}{\cos(x)}\,du = e^x + C$$

and the $\cos(x)$ bits cancel! Yippee!!

$$2\int u^2\,du = e^x + C$$

so we can now integrate the left hand side

$$\begin{aligned} 2\cdot\tfrac{1}{3}u^3 &= e^x + C \\ \tfrac{2}{3}\sin^3(y) &= e^x + C \end{aligned}$$

Now we are told that $y = \frac{\pi}{6}$ at $x = 0$, so

$$\begin{aligned} \tfrac{2}{3}\sin^3\left(\tfrac{\pi}{6}\right) &= e^0 + C \\ \tfrac{2}{3}\cdot\tfrac{1}{8} &= 1 + C \\ \tfrac{1}{12} - 1 &= C \\ C &= -\tfrac{11}{12} \end{aligned}$$

so our solution is

$$\tfrac{2}{3}\sin^3(y) = e^x - \tfrac{11}{12}$$

Tips, Tricks and Patterns

Patterns:

$$\int f'(x)\cdot[f(x)]^n\,dx, \qquad \int f'(x)\cdot e^{f(x)}\,dx, \qquad \int \frac{f'(x)}{[f(x)]^n}\,dx \qquad \text{indeed...} \qquad \int f'(x)\cdot g[f(x)]\,dx$$

Question 7

The Question

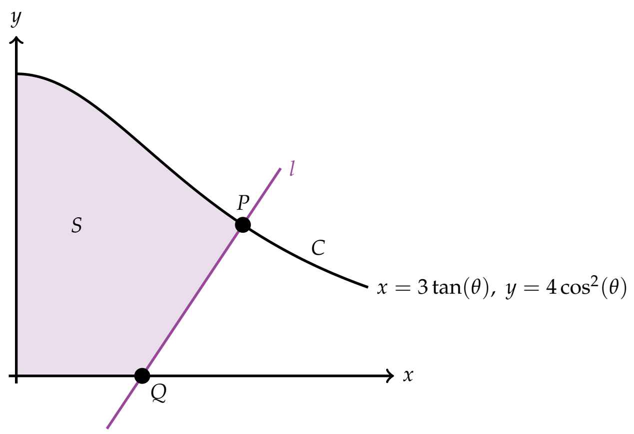

Figure 54: A graph of $x = 3\tan(\theta),\ y = 4\cos^2(\theta)$

Figure 54 shows a sketch of part of the curve C with parametric equations

$$x = 3\tan(\theta), \qquad y = 4\cos^2(\theta), \qquad 0 \leq \theta < \tfrac{\pi}{2}$$

The point P lies on C and has coordinates $(3, 2)$.

The line l is the normal to C at P. The normal cuts the x-axis at the point Q.

(a) Find the x-coordinate of the point Q. **(6 marks)**

The finite region S, shown shaded in Figure 54, is bounded by the curve C, the x-axis, the y-axis and the line l. This shaded region is rotated 2π radians about the x-axis to form a solid of revolution.

(b) Find the exact value of the volume of the solid of revolution, giving your answer in the form $p\pi + q\pi^2$, where p and q are rational numbers to be determined. **(9 marks)**

[You may use the formula $V = \frac{1}{3}\pi r^2 h$ for the volume of a cone.]

My Answer

(a) Q turns out to be at $\left(\frac{5}{3}, 0\right)$.

(b) Have a look at Figure 55. What we're going to have to do is to find the volume of revolution of the rotated area $S + R$, and then subtract the volume of revolution of the rotated region R. To find the volume of revolution of $S + R$ we will need this integral

$$V_{S+R} = \pi \int y^2\,dx$$

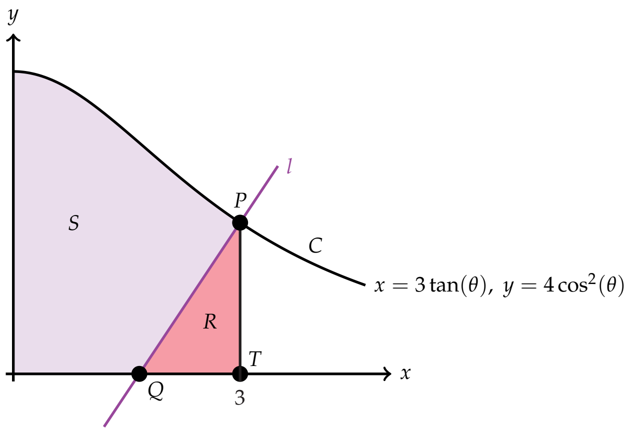

Figure 55: A graph of $x = 3\tan(\theta),\ y = 4\cos^2(\theta)$

And we've got to put everything in the integral in terms of θ. The y is easy: that's just going to be $4\cos^2(\theta)$. To find the dx in terms of $d\theta$ we must differentiate the $x = 3\tan(\theta)$ equation:

$$\begin{aligned} x &= 3\tan(\theta) \\ \frac{dx}{d\theta} &= 3\sec^2(\theta) \\ dx &= 3\sec^2(\theta)\ d\theta \end{aligned}$$

So, plugging this stuff in we get

$$\begin{aligned} V_{S+R} &= \pi \int_{x=0}^{x=3} (4\cos^2(\theta))^2 \cdot 3\sec^2(\theta)\ d\theta \\ &= \pi \int_{x=0}^{x=3} 48\cos^4(\theta)\sec^2(\theta)\ d\theta \\ &= 24\pi \int_{x=0}^{x=3} 2\cos^2(\theta)\ d\theta \end{aligned}$$

Now we've come across integrating $\cos^2(x)$ before. We had to use the world famous trigonometrical identity $\cos(2x) \equiv 2\cos^2(x) - 1$ to give $2\cos^2(x) \equiv 1 + \cos(2x)$. So, sticking this in:

$$V_{S+R} = 24\pi \int_{x=0}^{x=3} 1 + \cos(2\theta)\ d\theta$$

And at last we have a standard form that we can integrate. But before that, we have to do something with the limits. Now, when $x = 0$, $\theta = 0$, and when $x = 3$, $\theta = \frac{\pi}{4}$, so

$$\begin{aligned} V_{S+R} &= 24\pi \int_{\theta=0}^{\theta=\frac{\pi}{4}} 1 + \cos(2\theta)\ d\theta \\ &= 24\pi\Big[\theta + \tfrac{1}{2}\sin(2\theta)\Big]_{\theta=0}^{\theta=\frac{\pi}{4}} \\ &= 24\pi\Big[\Big\{\tfrac{\pi}{4} + \tfrac{1}{2}\sin(2\cdot\tfrac{\pi}{4})\Big\} - \Big\{0 + \tfrac{1}{2}\sin(2\cdot 0)\Big\}\Big] \\ &= 24\pi\Big[\tfrac{\pi}{4} + \tfrac{1}{2}\Big] \\ &= 6\pi^2 + 12\pi \end{aligned}$$

Right, now to find the rotated volume R, we could do the volume of revolution thing for the line l between the limits $x = \frac{5}{3}$ and $x = 3$. Or we could take the hint: this volume will be a *cone*, of radius 2 (the distance PT), and height $3 - \frac{5}{3} = \frac{4}{3}$ (the distance QT).

So,

$$\begin{aligned} V_R &= \tfrac{1}{3}\pi r^2 h \\ V_R &= \tfrac{1}{3} \cdot \pi \cdot 2^2 \cdot \tfrac{4}{3} \\ V_R &= \tfrac{16}{9}\pi \end{aligned}$$

So the volume we want will be

$$\begin{aligned} V_{S+R} - V_R &= 6\pi^2 + 12\pi - \tfrac{16}{9}\pi \\ &= 6\pi^2 + \tfrac{92}{9}\pi \end{aligned}$$

Tips, Tricks and Patterns

Tips: When you have $\sin^2(x)$ or $\cos^2(x)$ to integrate, there is a trick.

You have to use the identities: $\boldsymbol{\cos(2x) \equiv 2\cos^2(x) - 1 \equiv 1 - 2\sin^2(x)}$.

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Question 2

The Question

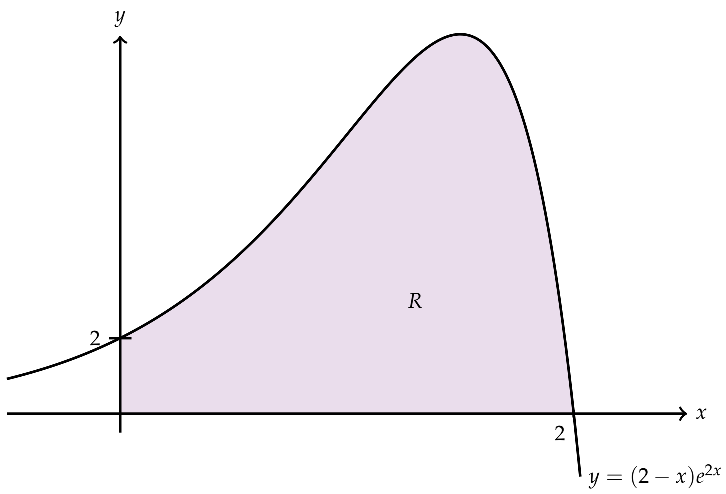

Figure 56: A graph of $y = (2-x)e^{2x}$

Figure 56 shows a sketch of part of the curve with equation

$$y = (2-x)e^{2x}, \qquad x \in \mathbb{R}$$

The finite region R, shown shaded in Figure 56, is bounded by the curve, the x-axis and the y-axis.

(a) ...

(b) ...

(c) Use calculus, showing each step in your working, to obtain an exact value for the area of R. Give your answer in its simplest form. **(5 marks)**

My Answer

(c) The integral we want will be

$$R = \int_{x=0}^{x=2} (2-x)e^{2x}\,dx$$

Well, this is a classic integration by parts. Whenever we have a polynomial multiplied by a sine, cosine, exponential or log, it's parts. So...check out Figure 57 for DIS is how we do it!

D	I	S
$2-x$	e^{2x}	$+$
-1	$\frac{1}{2}e^{2x}$	$-$
0	$\frac{1}{4}e^{2x}$	$+$
		$-$

Figure 57: Integrating $\int (2-x)e^{2x}\,dx$

So our integral becomes

$$R = \left[\tfrac{1}{2}(2-x)e^{2x} + \tfrac{1}{4}e^{2x}\right]_{x=0}^{x=2}$$

Now we can plug the numbers in:

$$\begin{aligned} R &= \left[\left\{\tfrac{1}{2}(2-2)e^{2\cdot 2} + \tfrac{1}{4}e^{2\cdot 2}\right\} - \left\{\tfrac{1}{2}(2-0)e^{2\cdot 0} + \tfrac{1}{4}e^{2\cdot 0}\right\}\right] \\ &= \left[\left\{0 + \tfrac{1}{4}e^{4}\right\} - \left\{1 + \tfrac{1}{4}\right\}\right] \\ &= \tfrac{1}{4}e^4 - \tfrac{5}{4} \\ &= \tfrac{1}{4}\left[e^4 - 5\right] \end{aligned}$$

Question 4

The Question

(a) Express

$$\frac{25}{x^2(2x+1)}$$

in partial fractions. **(4 marks)**

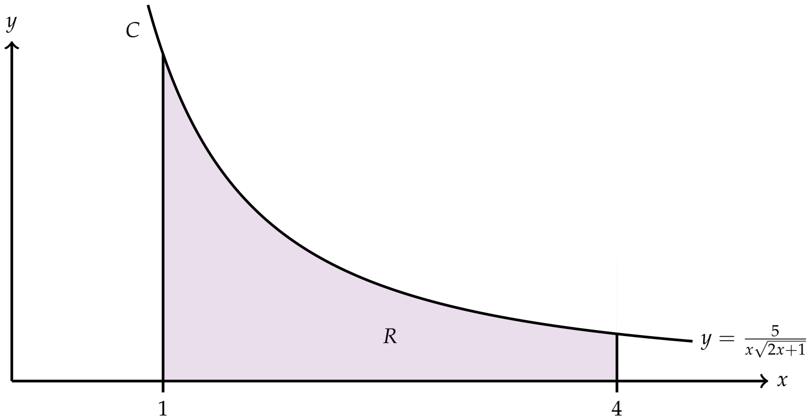

Figure 58: A graph of $y = \frac{5}{x\sqrt{2x+1}}$

Figure 58 shows a sketch of part of the curve C with equation

$$y = \frac{5}{x\sqrt{2x+1}}, \qquad x > 0$$

The finite region R is bounded by the curve C, the x-axis, the line with equation $x = 1$ and the line with equation $x = 4$. This region is shown shaded in Figure 58. The region R is rotated through $360°$ about the x-axis.

(b) Use calculus to find the exact volume of the solid of revolution generated, giving your answer in the form $a + b\ln(c)$, where a, b and c are constants. **(6 marks)**

My Answer

(a) This turns out to be

$$\frac{25}{x^2(2x+1)} = -\frac{50}{x} + \frac{25}{x^2} + \frac{100}{2x+1}$$

(b) The integral we want will be

$$\begin{aligned} V &= \pi \int y^2 \, dx \\ &= \pi \int_{x=1}^{x=4} \frac{25}{x^2(2x-1)} \, dx \\ &= \pi \int_{x=1}^{x=4} -\frac{50}{x} + \frac{25}{x^2} + \frac{100}{2x+1} \, dx \end{aligned}$$

using the answer to part (a). If I write these as separate integrals I get

$$V = -50\pi \int_{x=1}^{x=4} \frac{1}{x}\, dx + 25\pi \int_{x=1}^{x=4} x^{-2}\, dx + 50\pi \int_{x=1}^{x=4} \frac{2}{2x+1}\, dx$$

taking care in how I write them! The first two integrals are now standard forms; the third is one of these

$$\int \frac{f'(x)}{f(x)}\, dx = \ln\left[f(x)\right]$$

so I can now go ahead an integrate everything:

$$\begin{aligned} V &= -50\pi\Big[\ln(x)\Big]_{x=1}^{x=4} + 25\pi\left[-\frac{1}{x}\right]_{x=1}^{x=4} + 50\pi\Big[\ln(2x+1)\Big]_{x=1}^{x=4} \\ &= 25\pi\left[-2\ln(x) - \frac{1}{x} + 2\ln(2x+1)\right]_{x=1}^{x=4} \end{aligned}$$

Then plug the numbers in:

$$\begin{aligned} V &= 25\pi\left[\left\{-2\ln(4) - \tfrac{1}{4} + 2\ln(2\cdot 4+1)\right\} - \left\{-2\ln(1) - \tfrac{1}{1} + 2\ln(2\cdot 1+1)\right\}\right] \\ &= 25\pi\left[\left\{-2\ln(4) - \tfrac{1}{4} + 2\ln(9)\right\} - \{0 - 1 + 2\ln(3)\}\right] \\ &= 25\pi\left[2\ln\left(\tfrac{9}{4}\right) - \tfrac{1}{4} + 1 - 2\ln(3)\right] \\ &= 25\pi\left[2\ln\left(\tfrac{9}{12}\right) + \tfrac{3}{4}\right] \\ &= 25\pi\left[2\ln\left(\tfrac{3}{4}\right) + \tfrac{3}{4}\right] \end{aligned}$$

Tips, Tricks and Patterns

Patterns:

$$\int f'(x)\cdot\left[f(x)\right]^n\, dx, \qquad \int f'(x)\cdot e^{f(x)}\, dx, \qquad \int \frac{f'(x)}{\left[f(x)\right]^n}\, dx \qquad \text{indeed...} \qquad \int f'(x)\cdot g\left[f(x)\right]\, dx$$

Question 7

The Question

The rate of increase of the number, N, of fish in a lake is modelled by the differential equation

$$\frac{dN}{dt} = \frac{(kt-1)(5000-N)}{t}, \qquad t > 0,\ 0 < N < 5000$$

In the given equation, the time t is measured in years from the start of January 2000 and k is a positive constant.

(a) By solving the differential equation, show that

$$N = 5000 - Ate^{-kt}$$

where A is a positive constant. **(5 marks)**

After one year, at the start of January 2001, there are 1200 fish in the lake.

After two years, at the start of January 2002, there are 1800 fish in the lake.

(b) Find the exact value of the constant A and the exact value of the constant k. **(4 marks)**

(c) Hence find the number of fish in the lake after five years. Give your answer to the nearest hundred fish. **(1 mark)**

My Answer

(a) The only way we know how to solve differential equations at A-Level is to separate the variables

$$\begin{aligned} \frac{1}{5000-N}\, dN &= \frac{kt-1}{t}\, dt \\ &= k - \frac{1}{t}\, dt \end{aligned}$$

and integrate both sides

$$\int \frac{1}{5000-N}\, dN = \int k - \frac{1}{t}\, dt$$

The right hand side is a pair of standard forms; I recognise the left hand side now: it's one of these $\int \frac{f'(x)}{f(x)}\,dx = \ln\left[f(x)\right]$ or at least it would be if I muliplied both sides by -1:

$$\int \frac{-1}{5000-N}\,dN = \int \frac{1}{t} - k\,dt$$

Ha! So now we can integrate:

$$\ln(5000-N) = \ln(t) - kt + \ln(C)$$

again using $\ln(C)$ when I have a log on the left. Now we need to make N the subject of this:

$$\ln\left(\frac{5000-N}{C}\right) = \ln(t) - kt$$
$$\frac{5000-N}{C} = e^{\ln(t)-kt}$$

Now on the right hand side we can use the way indices work to write it like this

$$\frac{5000-N}{C} = e^{\ln(t)}\cdot e^{-kt}$$
$$= te^{-kt}$$

So,

$$5000-N = Cte^{-kt}$$
$$N = 5000 - Cte^{-kt}$$

as required.

(b) Now we are told that $N = 1200$ when $t = 1$:

$$1200 = 5000 - C\cdot 1\cdot e^{-k\cdot 1}$$
$$Ce^{-k} = 3800$$

Now we are told that $N = 1800$ when $t = 2$:

$$1800 = 5000 - C\cdot 2\cdot e^{-k\cdot 2}$$
$$2Ce^{-2k} = 3200$$
$$Ce^{-2k} = 1600$$

Dividing these equations we get

$$\frac{Ce^{-k}}{Ce^{-2k}} = \frac{3800}{1600}$$
$$\frac{e^{-k}}{e^{-2k}} = \frac{38}{16}$$
$$e^{k} = \frac{19}{8}$$
$$k = \ln\left(\frac{19}{8}\right)$$

Putting this value of k into one of our equations gives

$$Ce^{-\ln\left(\frac{19}{8}\right)} = 3800$$
$$Ce^{\ln\left(\frac{8}{19}\right)} = 3800$$
$$C\cdot\tfrac{8}{19} = 3800$$
$$C = 3800\cdot\tfrac{19}{8}$$
$$C = 9025$$

(c), So, when $t = 5$,

$$N = 5000 - 9025\cdot 5\cdot e^{-\ln\left(\frac{19}{8}\right)\cdot 5}$$
$$= 5000 - 45125\cdot e^{\ln\left(\frac{8}{19}\right)\cdot 5}$$
$$= 5000 - 45125\left\{e^{\ln\left(\frac{8}{19}\right)}\right\}^{5}$$
$$= 5000 - 45125\cdot\left(\tfrac{8}{19}\right)^{5}$$
$$= 4400 \text{ fish, to the nearest hundred.}$$

Tips, Tricks and Patterns

Patterns:

$$\int f'(x)\cdot\left[f(x)\right]^n\,dx, \qquad \int f'(x)\cdot e^{f(x)}\,dx, \qquad \int \frac{f'(x)}{\left[f(x)\right]^n}\,dx \qquad \text{indeed...} \qquad \int f'(x)\cdot g\left[f(x)\right]\,dx$$

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A Standard Forms

Here is a list of standard forms. It's not a comprehensive list, but it includes some important integrals that you either have to know (the top half of the table), or will be in your formula booklet (the bottom half).

Function	**Integral**
ax^n	$\frac{a}{n+1}x^{n+1}$
$\sin(x)$	$-\cos(x)$
$\cos(x)$	$\sin(x)$
e^x	e^x
$\ln(x)$	$\frac{1}{x}$
a^{bx}	$\frac{a^{bx}}{b\ln(a)}$
$\sec^2(kx)$	$\frac{1}{k}\tan(kx)$
$\sec(x)\ \tan(x)$	$\sec(x)$
$\tan(x)$	$\ln(\|\sec(x)\|)$
$\cot(x)$	$\ln(\|\sin(x)\|)$
$\text{cosec}(x)$	$-\ln(\|\text{cosec}(x)+\cot(x)\|)$
$\sec(x)$	$\ln(\|\sec(x)+\tan(x)\|)$
$-\text{cosec}^2(x)$	$\cot(x)$
$-\text{cosec}(x)\ \cot(x)$	$\text{cosec}(x)$

Table 1: Integral Standard Forms

B Differentiating and Integrating a^x

B.1 Differentiating a^x

If you have something like this to differentiate

$$y = a^x$$

where a is a constant, you can't use any of the standard rules that we know and love. You can't use the $y = ax^n$ polynomial rule because this time the base is a constant, and the power is the variable (it's the other way around in the ploynomial rule). You can't use the chain rule, the product rule or the quotient rule. So what can we do?

Well the problem is the fact that the x is a power. We have to get it down from being a power. And how do we do that? There is a standard technique: we take logs of both sides! Next problem: which log? Well, we know that we will be differentiating this at some point, and the most convenient log for calculus is $\ln(x)$. So let's take the natural log of both sides:

$$\begin{aligned}\ln(y) &= \ln\left(a^x\right) \\ &= x\ln(a)\end{aligned}$$

Now we come to the differentiating bit: we differentiate both sides with respect to x, *implicitly*:

$$\begin{aligned}\frac{d}{dx}\Big[\ln(y)\Big] &= \frac{d}{dx}\Big[x\ln(a)\Big] \\ \frac{1}{y}\frac{dy}{dx} &= \ln(a)\end{aligned}$$

remembering that $\ln(a)$ is just a constant, so differentiating $x\ln(a)$ is a bit like differentiating $3x$. So:

$$\begin{aligned}\frac{dy}{dx} &= y\ln(a) \\ &= \ln(a)a^x\end{aligned}$$

Now, we can use other rules (chain, product, quotient) in combination with this standard form, of course: so we have to bear in mind that we might need to differentiate things like

$$y = a^{bx}$$

for example. And how would we do this one? Well, just do the same thing we do for all chain rule differentiations: we use a substitution, followed by the chain rule itself:

$$\text{Let} \qquad u = bx \qquad \text{then} \qquad \frac{du}{dx} = b$$

so our function would become

$$y = a^u$$

and we now know how to differentiate this:

$$\frac{dy}{du} = \ln(a)a^u$$

The problem here of course is that we don't want $\frac{dy}{du}$. We want $\frac{dy}{dx}$. But we can get it using the chain rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= a^u\ln(a) \times b \\ &= b\ln(a)a^{bx}\end{aligned}$$

B.2 Integrating a^x

Now why is this result important for a document about integration? Well, since differentiation and integration are inverse processes, then

$$\text{If} \qquad y = a^{bx} \qquad \text{then} \qquad \frac{dy}{dx} = b\ln(a)a^{bx}$$

would mean that

$$\text{If} \qquad y = a^{bx} \qquad \text{then} \qquad \int y\ dx = \frac{a^{bx}}{b\ln(a)}$$

And this is a result that you should learn, as it isn't in your formula book. That's why I've included it in the top section of my standard forms list (Appendix A).

References

Smith, S. (2016). How To Do Integration I: The Process.