Steve Smith Tuition: Maths Notes

How To Do Integration I: The Process

Contents

Prerequisites

Knowledge of the four techniques of integral transformation is required!!

Notes

None.

Document History

Date	Version	Comments
6^{th} March 2016	1.0	Initial creation of the document.
10^{th} March 2016	1.1	Minor tweeks to the flowchart after some complaints...

1 Introduction

Most students find integration difficult. Not only are there specific techniques (such as integration by parts or substitution) to master, but there is also the issue of "Which technique do I use to integrate this thing?".

Sometimes you are lucky. Sometimes a question tells you what technique to use. For example, Edexcel GCE C4 June 2005 Question 4 asks[1]:

Use the substitution $x = \sin(\theta)$ to find the exact value of

$$\int_{x=0}^{x=\frac{1}{2}} \frac{1}{\left[1-x^2\right]^{\frac{3}{2}}}\, dx$$

Well, we're going to use substitution, then.

But other times you are just given an integral to do and you have no clue what method to use to integrate it. Then what?

This document is meant to shed some light on *then what*? But first, a quick overview of...

1.1 Integration as a Whole

You can only integrate what I call *standard forms*. These are functions that you just know the integrals of. They are primarily things like: polynomial terms ax^n; trigonometrical functions like $\sin(x)$ and $\cos(x)$; and other assorted things like e^x and $\ln(x)$. See Appendix A for a more extensive list of standard forms.

These days you are given a formula sheet in an A-Level maths exam, and in the formula sheet will be a list of functions and their integrals. In that case, everything in the list will be a standard form, as if you needed to integrate a function in the list, you could just write down the answer. There's no working to do.

So, what happens when you get an integral that's *not* a standard form, and you're *not* told how to do it?

Well, there's only *four* things you can do[2]. Each one of these four techniques is used *to transform your integral into a standard form*. The four techniques are:

- partial fractions;
- use of trigonometrical identities;
- substitution;
- integration by parts.

That's all. Only those four. Nothing else.

I've written some documents looking specifically at integration by parts [Smith (2012a) and Smith (2012b)] and integration by substitution Smith (2016b). You might want to go and have a look at those before you carry on. From here on in, I'm going to assume that you know what to do once you have decided on a particular technique, for each of the techniques listed above.

1.2 Purpose

The purpose of this document is to show my thought processes when I tackle integrations. Here, I'm hoping to show you how you decide what technique to use to solve integrals.

I've also done lots of examples from past papers. I've chosen to tackle all the Edexcel C4 past papers from 2005 onwards. Why Edexcel? Well, I had to choose something! All these questions and my worked answers can be found in Smith (2016a).

[1] Actually I have not reproduced this question accurately. I have asked the question how it should have been written. More of this as you read on...

[2] I'm talking here about *algaebraic* integration. You could always use *numerical* integration to integrate *any* function, even standard forms. That's a subject for another day...

2 Mark Breakdown By Paper

As I went along answering the Edexcel C4 past paper integration questions, I made a list of what kinds of integrals come up, and how many marks are directly attributable to each type. Here are my results.

In the following table, headers represent the following type of question

- **SF** : Standard form. Something that you either know, or is in the formula book.
- **PF** : Partial Fractions.
- **TI** : Use of trigonometrical identities.
- **Sub** : Substitutions of various kinds.
- **Parts** : Integration by Parts, which I suggest you use *DIS* for.
- **Obtain** : Coming up with the integral or differential equation to solve.
- **Evaluate** : Plugging the numbers into definite integrals that you have already solved.

Paper	SF	PF	TI	Sub	Parts	Obtain	Evaluate	Total
Jun 2005		8		13	5		5	31
Jan 2006			7	12	8	5		32
Jun 2006	7		6	3	6			22
Jan 2007		8		10	5		4	27
Jun 2007	8	10	3	10	4		3	38
Jan 2008		6	5	11	4	9	2	37
Jun 2008		11		4	6	4	2	27
Jan 2009			2	16	4			22
Jun 2009		7	5	5	4	5	5	31
Jan 2010	14			7	4	3	6	34
Jun 2010			3	12	7			22
Jan 2011	6	12		8	6			32
Jun 2011	6		7	6	6			25
Jan 2012		11		10	6		1	28
Jun 2012	5	10			4		3	22
Jan 2013	6		6	12	5		12	41
Jun 2013		7		15	5		5	32
Jun 2013 R	5			14	6	3	4	32
Jun 2014			16	8	3		6	33
Jun 2014 R		10		5	5		5	25

Table 1: Mark Breakdown By Paper

So, on average, between June 2005 and June 2014 (inclusive), the percentage of marks directly attributable to integration questions (to the nearest percent) in Edexcel C4 exams is

$$\frac{593}{20 \times 75} \times 100 = 40\%$$

Other exam boards have similar figures.

3 Integral Breakdown By Type

Here is a rough breakdown on the types of techniques that were tested on the Edexcel papers.

What I mean by "unaided" substitutions is that in this type of integral a substitution is required, but you are not told what substitution to use.

What I mean by "aided" substitutions is that in this type of integral a substitution is required, and you are told what substitution to use.

What I mean by a "classic" substitution is this. When you learn about using substitution to solve integrals, there is a very common family of integrals that you learn about. These integrals are

$$\int f'(x) \cdot [f(x)]^n \, dx, \quad \int f'(x) \cdot e^{f(x)} \, dx, \quad \int \frac{f'(x)}{[f(x)]^n} \, dx, \quad \text{indeed...} \quad \int f'(x) \cdot g[f(x)] \, dx$$

where the coloured bits are the things to look out for. If you have an integral of one of these types, then you use substitution to transform the integral into a standard form, and the substitution is $u = f(x)$.

Substitutions			Other Types		Trig Identities	
Unaided	**Aided**	**Classic**	**Parts**	**Standard Forms**	$\sin^2(x)$ **etc.**	**Other**
12	13	22	23	11	7	2

Table 2: Integral Breakdown By Type

4 What Integrals Were Not Tested (!!)

I've had a look not only at the Edexcel specification and exams, but also those from the OCR-A and OCR-B (MEI) specifications. Here is a very interesting thing. It's a list of integrals that are in the various specifications or text books, but were never tested in the studied period.

- Any powers of $\sin(x)$ or $\cos(x)$ higher than 2: this has never been tested in any specification.
- Integrals of the types $\int \sin(ax)\sin(bx)\,dx$, $\int \sin(ax)\cos(bx)\,dx$ or $\int \cos(ax)\cos(bx)\,dx$: this is outside the scope of all specifications.
- Integrals of the types $\int \sec^n(x)\tan(x)\,dx$ or $\int \tan^n(x)\sec^2(x)\,dx$: this is outside the scope of all specifications.
- Integrals of the types $\int e^{ax}\sin(bx)\,dx$ or $\int e^{ax}\cos(bx)\,dx$: this is outside the scope of all specifications.

Solids of revolution where the function is rotated around the y-axis: this is *outside the scope of the Edexcel specification*, but both OCR-A and OCR-B expect you to be able to do these: questions have been set.

5 A-Level Integration Decision Flow Chart

So, bearing in mind the sorts of integrals that you need to be able to solve at A-Level, and indeed, the sorts of integrals that you are *not* required to solve at A-Level, I've put together the following process. You start at the "Start"; grey boxes represent tests on the integrand of your integral. Follow the arrows out of a test box depending on the result of the test.

Notice that we might have to go around the process more than once...

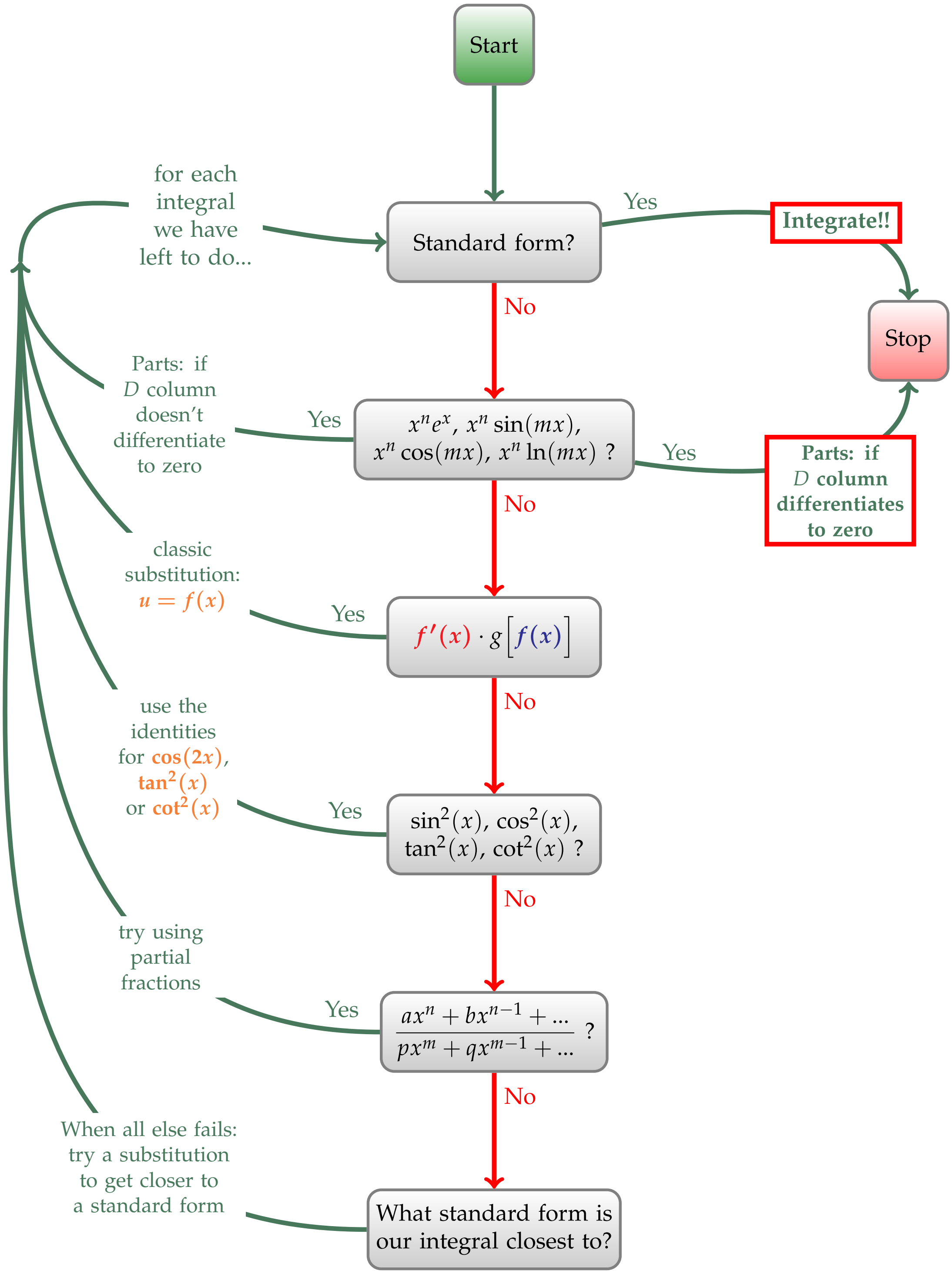

Figure 1: Integration Decision Flow Chart

The rest of this document shows you examples of the process at work.

6 Examples

6.1 A Standard Form

Let's say you had this integral to work through:

$$I = \int \frac{1}{x^2}\,dx$$

OK. So, let's work through the process.

First test: "is the integrand a standard form"? The answer to this question is "Yes": this integrand is of the form ax^n, so we can go to the "Integrate!!" step, just writing down the answer, once the integral is written in the right form:

$$\begin{aligned} I &= \int x^{-2}\,dx \\ &= \frac{x^{-1}}{-1} + C \\ &= -\frac{1}{x} + C \end{aligned}$$

and we are done.

6.2 Integration by Parts

6.2.1 When the D Column Differentiates to Zero

Let's say we needed to find

$$\int xe^{2x}\,dx$$

OK. So, let's work through the process.

First test: "is the integrand a standard form"? The answer to this question is "No".

Next test: "is the integral x^ne^x, $x^n\sin(mx)$, $x^n\cos(mx)$, or $x^n\ln(mx)$? "Yes" is the answer to this question. This is a classic integration by parts. Whenever we have a polynomial multiplied by a sine, cosine, exponential or log, it's parts. So...DIS is how we do it:

D	I	S
x	e^{2x}	$+$
1	$\frac{1}{2}e^{2x}$	$-$
0	$\frac{1}{4}e^{2x}$	$+$
		$-$

Figure 2: Integrating $\int xe^{2x}\,dx$

So that

$$\int xe^{2x}\,dx = \tfrac{1}{2}xe^{2x} - \tfrac{1}{4}e^{2x} + C$$

6.2.2 When the D Column *Doesn't* Differentiate to Zero

Let's say we needed to find

$$I = \int (x-1)\ln(x)\; dx$$

OK. So, let's work through the process.

First test: "is the integrand a standard form"? The answer to this question is "No".

Next test: "is the integral $x^n e^x$, $x^n \sin(mx)$, $x^n \cos(mx)$, or $x^n \ln(mx)$?" "Yes" is the answer to this question. This is a classic integration by parts. Whenever we have a polynomial multiplied by a sine, cosine, exponential or log, it's parts. So...DIS is how we do it. As we don't know how to integrate $\ln(x)$, that has to go in the D column:

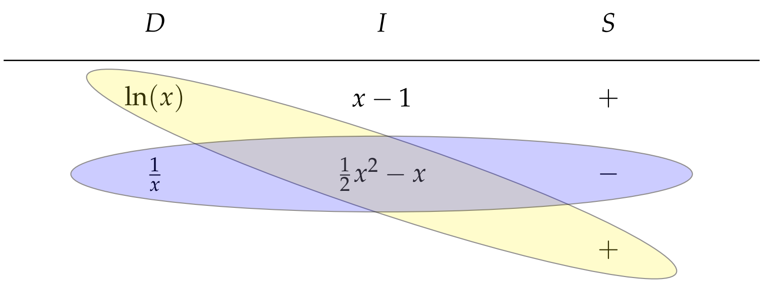

Figure 3: Integrating $\int (x-1)\ln(x)\; dx$

So our integral becomes

$$\begin{aligned} I &= \left(\tfrac{1}{2}x^2 - x\right)\ln(x) - \int \frac{1}{x}\cdot\left(\frac{1}{2}x^2 - x\right)\; dx \\ &= \left(\tfrac{1}{2}x^2 - x\right)\ln(x) - \int \tfrac{1}{2}x - 1\; dx \end{aligned}$$

Now from our process point of view, we still have an integral to solve. And that's because we didn't have a situation where the D column differentiated away to zero. So we have to go around the cycle again for the new integral. The new integral turns out to be a standard form, so we will be able to integrate it next time round the loop.

6.3 A "Classic" Substitution

Let's say we had

$$I = \int 2x\, e^{x^2}\; dx$$

to find. OK. So, let's work through the process.

First test: "is the integrand a standard form"? The answer to this question is "No".

Next test: "is the integrand $x^n e^x$, $x^n \sin(mx)$, $x^n \cos(mx)$, or $x^n \ln(mx)$?" The answer to this question is "No".

Next test: "is the integrand $\dfrac{f'(x)}{\left[f(x)\right]^n}$, $f'(x)\cdot\left[f(x)\right]^n$, $f'(x)\cdot e^{f(x)}$, ...etc?" The answer to this question is "Yes"! So, we use the substitution $u = f(x)$:

If $\quad u = x^2 \quad$ then $\quad \dfrac{du}{dx} = 2x \quad$ and $\quad dx = \dfrac{1}{2x}\, du$

Plugging this into our integral gives

$$I = \int 2x\, e^u \cdot \frac{1}{2x}\, du$$
$$= \int e^u\, du$$

and around the cycle we go again. Next time around, we will discover that we now have a standard form, so we will be able to integrate it.

6.4 Square of Triggy Thing

Let's say we had this integral

$$I = \int \cos^2(x)\, dx$$

to find. OK. So, let's work through the process.

First test: "is the integrand a standard form"? The answer to this question is "No".

Next test: "is the integrand $x^n e^x$, $x^n \sin(mx)$, $x^n \cos(mx)$, or $x^n \ln(mx)$?" The answer to this question is "No".

Next test: "is the integrand $\frac{f'(x)}{[f(x)]^n}$, $f'(x) \cdot [f(x)]^n$, $f'(x) \cdot e^{f(x)}$, ...etc?" The answer to this question is "No".

Next test: "is the integrand $\sin^2(x)$, $\cos^2(x)$, $\tan^2(x)$, $\cot^2(x)$? "Yes"!! Now in this case, we need to use a trigonometrical identity. Which one? Well, if you have

- $\sin^2(x)$, you want to use $\cos(2x) \equiv 1 - 2\sin^2(x)$, to convert the integral into one with a $\cos(2x)$ in it;
- $\cos^2(x)$, you want to use $\cos(2x) \equiv 2\cos^2(x) - 1$, to convert the integral into one with a $\cos(2x)$ in it;
- $\tan^2(x)$, you want to use $\tan^2(x) + 1 \equiv \sec^2(x)$, ... with a $\sec^2(x)$ in it;
- $\cot^2(x)$, you want to use $1 + \cot^2(x) \equiv \mathrm{cosec}^2(x)$, ... with a $\mathrm{cosec}^2(x)$ in it.

Why? because $\cos(x)$, $\sec^2(x)$ and $\mathrm{cosec}^2(x)$ are all standard forms.

So we can transform our integral into

$$I = \tfrac{1}{2} \int \cos(2x) + 1\, dx$$

and around the process we go again. Next time around, we might spot that we can integrate $\cos(2x)$. If you don't spot that, use the substitution $u = 2x$, and go around the cycle again...

6.5 Partial Fractions

Let's say we had this integral

$$\int \frac{5x+3}{(2x-3)(x+2)}\, dx$$

OK. So, let's work through the process.

First test: "is the integrand a standard form"? The answer to this question is "No".

Next test: "is the integrand $x^n e^x$, $x^n \sin(mx)$, $x^n \cos(mx)$, or $x^n \ln(mx)$?" The answer to this question is "No".

Next test: "is the integrand $\frac{f'(x)}{[f(x)]^n}$, $f'(x) \cdot [f(x)]^n$, $f'(x) \cdot e^{f(x)}$, ...etc?" The answer to this question is "No".

Next test: "is the integrand $\sin^2(x)$, $\cos^2(x)$, $\tan^2(x)$, $\cot^2(x)$?" The answer to this question is "No".

Next test: "is the integrand $\frac{\text{polynomial}}{(ax+b)(cx+d)\ldots}$?" The answer to this question is "Yes"!! So, let's see if we can use partial fractions here. It turns out that

$$\frac{5x+3}{(2x-3)(x+2)} = \frac{3}{2x-3} + \frac{1}{x+2}$$

so our integral becomes

$$\int \frac{3}{2x-3} + \frac{1}{x+2}\, dx$$

and around the cycle we go again, once for each new integral we have to find...

6.6 Unaided Substitution

Let's say we had

$$\begin{aligned} I &= \int \frac{1}{(2t+1)^2}\, dt \\ &= \int (2t+1)^{-2}\, dt \end{aligned}$$

to solve. OK. So, let's work through the process.

First test: "is the integrand a standard form"? The answer to this question is "No".

Next test: "is the integrand $x^n e^x$, $x^n \sin(mx)$, $x^n \cos(mx)$, or $x^n \ln(mx)$?" The answer to this question is "No".

Next test: "is the integrand $\frac{f'(x)}{[f(x)]^n}$, $f'(x) \cdot [f(x)]^n$, $f'(x) \cdot e^{f(x)}$, ...etc?" The answer to this question is "No".

Next test: "is the integrand $\sin^2(x)$, $\cos^2(x)$, $\tan^2(x)$, $\cot^2(x)$?" The answer to this question is "No".

Next test: "is the integrand $\frac{\text{polynomial}}{(ax+b)(cx+d)\ldots}$?" The answer to this question is "No". We're running out of options! The only thing left is to have a look at the integrand, and see what it most resembles. Now in this case, it looks a bit like x^{-2}. That gives me the idea to try a substitution:

$$\text{If} \qquad u = 2t+1 \qquad \text{then} \qquad \frac{du}{dt} = 2 \qquad \text{and} \qquad dt = \tfrac{1}{2}\, du$$

Throwing this stuff into the integral gives

$$\begin{aligned} I &= \int u^{-2} \cdot \tfrac{1}{2}\, du \\ &= \tfrac{1}{2} \int u^{-2}\, du \end{aligned}$$

and around the cycle we go again.

Each time around the cycle, we are transforming our integral so that it gets closer to a standard form.

A Standard Forms

Here is a list of standard forms. It's not a comprehensive list, but it includes some important integrals that you either have to know (the top half of the table), or will be in your formula booklet (the bottom half).

Function	Integral
ax^n	$\frac{a}{n+1}x^{n+1}$
$\sin(x)$	$-\cos(x)$
$\cos(x)$	$\sin(x)$
e^x	e^x
$\ln(x)$	$\frac{1}{x}$
a^{bx}	$\frac{a^{bx}}{b\ln(a)}$
$\sec^2(kx)$	$\frac{1}{k}\tan(kx)$
$\sec(x)\ \tan(x)$	$\sec(x)$
$\tan(x)$	$\ln(\|\sec(x)\|)$
$\cot(x)$	$\ln(\|\sin(x)\|)$
$\text{cosec}(x)$	$-\ln(\|\text{cosec}(x)+\cot(x)\|)$
$\sec(x)$	$\ln(\|\sec(x)+\tan(x)\|)$
$-\text{cosec}^2(x)$	$\cot(x)$
$-\text{cosec}(x)\ \cot(x)$	$\text{cosec}(x)$

Table 3: Integral Standard Forms

References

Smith, S. (2012a). Integration by Parts, the Tabular Method I: "DIS is how you do it!". How to do integration by parts the easy way.

Smith, S. (2012b). Integration by Parts, the Tabular Method II: "DIS is how you do more with it!". How to do even more integrations by parts the easy way.

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