Steve Smith Tuition: Maths Notes

How to Calculate Mortgage Payments

Contents

Prerequisites

Knowledge of geometric series would be useful!

Notes

None.

Document History

Date	Version	Comments
5th January 2016	1.0	Initial creation of the document.
9th January 2016	1.1	Corrections and extensions...

1 Introduction

A mortgage[1] is a loan that you can take out to buy a property or land. As the price of the property is usually very great, it would be impossible for the ordinary person to buy it without a special kind of loan, one that you have to repay over a *very* long time. Hence, the mortgage. And you do pay them back over a *very* long time. 25 years is typical. The agreed length of a mortgage is known as the *term* of the mortgage.

Mortgages are typically the most expensive loan that ordinary people take out in the course of their lifetime[2].

When it came to buying my first property, there was no internet, and so if you wanted to find out what monthly mortgage payments would be if you took out a loan for a given term at a given annual interest rate, you had to ask the lender (face to face) what the payments would be. That would be annoying (for everyone) because I would want to know lots of options. And the guy in the Building Society wouldn't know how to calculate the thing, he would have to look it up in a table. But the tables were only printed giving the payments for the current interest rate offered by that particular lender. And of course, they varied considerably from lender to lender. So that meant a lot of leg work in comparing offers from lenders.

Wouldn't it be nice, I thought, if it was possible to come up with a formula to calculate monthly mortgage payments?

1.1 The Basics

The basics of any loan are these:

- The loan is *increased* by *interest* being added to the amount borrowed. This could occur just once, but typically interest is added frequently. Interest could be added annually, monthly, or even daily! Interest is the price you have to pay for taking out the loan.
- The loan is *reduced* by payments you make to the lender. These occur frequently too (possibly a bit too frequently), usually once every month.

The idea is that as time goes by your payments are more than the interest that's being added! Otherwise you will never pay the thing back.

So, without further ado, let's start looking at possible mortgage models...

[1]From two old french words: *gage*, meaning a *pledge*, and *mort*, meaning *death*. So a mortgage is literally *a pledge to the death*. Presumably not many people got to the end of their mortgage terms in those days...

[2]Unless you have a Wonga loan, of course.

2 Interest Added Yearly

We borrow £A at an interest rate of r per year[3] over a period of n years. Mortgage payments of £p occur every month. Banks and Building Societies being what they are, let's assume that interest is added at the *start* of each year, *before* any of the payments have been deducted from the outstanding debt.

The First Year

At the beginning of year 1, we borrow A. This our current outstanding debt, D_0.

At the start of year 1, we add the interest, and then deduct the twelve monthly payments at the end of the year. The interest I on the initial loan will be

$$I = A \times r$$

so at the end of year 1, the outstanding debt will be

$$\begin{aligned} D_1 &= A + A \times r - 12p \\ &= A(1+r) - 12p \end{aligned}$$

Notice that adding the interest at a rate r is equivalent to multiplying the outstanding debt by $(1+r)$.

The Second Year

At the beginning of year 2, the outstanding debt is $A(1+r) - 12p$.

At the start of year 2, we add the interest (equivalent to $D_1 \times (1+r)$) and subtract the twelve monthly payments at the end of the year to get the outstanding debt at the end of the year:

$$\begin{aligned} D_2 &= [A(1+r) - 12p](1+r) - 12p \\ &= A(1+r)^2 - 12p(1+r) - 12p \end{aligned}$$

The Third Year

At the beginning of year 3, the outstanding debt is $A(1+r)^2 - 12p(1+r) - 12p$.

At the start of year 3, we add the interest (equivalent to $D_2 \times (1+r)$) and subtract the twelve monthly payments at the end of the year, so at the end of year 3, the outstanding debt will be

$$\begin{aligned} D_3 &= [A(1+r)^2 - 12p(1+r) - 12p](1+r) - 12p \\ &= A(1+r)^3 - 12p(1+r)^2 - 12p(1+r) - 12p \end{aligned}$$

The Fourth Year

At the beginning of year 4, the outstanding debt is $A(1+r)^3 - 12p(1+r)^2 - 12p(1+r) - 12p$.

At the start of year 4, we add the interest (equivalent to $D_3 \times (1+r)$) and subtract the twelve monthly payments at the end of the year, so at the end of year 4, the outstanding debt will be

$$\begin{aligned} D_4 &= [A(1+r)^3 - 12p(1+r)^2 - 12p(1+r) - 12p](1+r) - 12p \\ &= A(1+r)^4 - 12p(1+r)^3 - 12p(1+r)^2 - 12p(1+r) - 12p \end{aligned}$$

[3]Here, r will be the interest rate percentage expressed as a decimal value. So for example, 4% would give $r = 0.04$.

The n^{th} Year

Can you see the pattern?

At the end of year n, the outstanding debt D_n will be

$$D_n = A(1+r)^n - 12p(1+r)^{n-1} - 12p(1+r)^{n-2} \ldots - 12p(1+r)^2 - 12p(1+r) - 12p$$

Now we can write this as

$$D_n = A(1+r)^n - 12p[1 + (1+r) + (1+r)^2 + \ldots + (1+r)^{n-2} + (1+r)^{n-1}]$$

But hey! The bit in the square brackets is a geometric series, the first term being 1, the ratio being $(1+r)$ and the number of terms being n. There is a formula for adding up the terms of a geometric series (see Appendix A) which we can use to add these terms up. So the sum of the series, S_n will be

$$\begin{aligned} S_n &= 1 + (1+r) + (1+r)^2 + \ldots + (1+r)^{n-2} + (1+r)^{n-1} \\ &= \frac{1 \times [(1+r)^n - 1]}{(1+r) - 1} \\ &= \frac{(1+r)^n - 1}{r} \end{aligned}$$

So after n years, our outstanding debt will be

$$D_n = A(1+r)^n - 12p\left[\frac{(1+r)^n - 1}{r}\right]$$

Now if n represents the term of our mortgage, then after n years we should have paid off the mortgage! So our outstanding debt should be 0, and so

$$A(1+r)^n = 12p\left[\frac{(1+r)^n - 1}{r}\right]$$

so that

$$p = \frac{rA(1+r)^n}{12[(1+r)^n - 1]}$$

so, finally,

$$p = \frac{r(1+r)^n}{12[(1+r)^n - 1]}A \qquad (1)$$

3 Interest Added Monthly

We borrow £A at an interest rate of r per *year* over a period of m months. Now if the interest rate is r per year, then the interest rate *per month* will be $\frac{r}{12}$. Mortgage payments of £p occur every month. Banks and Building Societies being what they are, let's assume that interest is added at the *start* of each month, *before* the payment for the month has been deducted from the outstanding debt.

The First Month

At the beginning of month 1, we borrow A. This our current outstanding debt, D_0.

At the start of month 1, we add the interest, and then deduct the monthly payment at the end of the month. So at the end of month 1, the outstanding debt will be

$$\begin{aligned} D_1 &= A + A \times \frac{r}{12} - p \\ &= A\left(1 + \frac{r}{12}\right) - p \end{aligned}$$

The Second Month

At the beginning of month 2, the outstanding debt is $A\left(1 + \frac{r}{12}\right) - p$.

At the start of month 2, we add the interest (equivalent to $D_1 \times \left(1 + \frac{r}{12}\right)$) and subtract the monthly payment at the end of the month to get the outstanding debt at the end of the month:

$$\begin{aligned} D_2 &= \left[A\left(1 + \frac{r}{12}\right) - p\right]\left(1 + \frac{r}{12}\right) - p \\ &= A\left(1 + \frac{r}{12}\right)^2 - p\left(1 + \frac{r}{12}\right) - p \end{aligned}$$

The Third Month

At the beginning of month 3, the outstanding debt is $A\left(1 + \frac{r}{12}\right)^2 - p\left(1 + \frac{r}{12}\right) - p$.

At the start of month 3, we add the interest (equivalent to $D_2 \times \left(1 + \frac{r}{12}\right)$) and subtract the monthly payment at the end of the month, so at the end of month 3, the outstanding debt will be

$$\begin{aligned} D_3 &= \left[A\left(1 + \frac{r}{12}\right)^2 - p\left(1 + \frac{r}{12}\right) - p\right]\left(1 + \frac{r}{12}\right) - p \\ &= A\left(1 + \frac{r}{12}\right)^3 - p\left(1 + \frac{r}{12}\right)^2 - p\left(1 + \frac{r}{12}\right) - p \end{aligned}$$

The m^{th} Month

Can you see the pattern?

At the end of month m, the outstanding debt D_m will be

$$D_m = A(1+r)^m - p\left(1 + \frac{r}{12}\right)^{m-1} - p\left(1 + \frac{r}{12}\right)^{m-2} ... - p\left(1 + \frac{r}{12}\right)^2 - p\left(1 + \frac{r}{12}\right) - p$$

Now we can write this as

$$D_m = A\left(1 + \frac{r}{12}\right)^m - p\left[1 + \left(1 + \frac{r}{12}\right) + \left(1 + \frac{r}{12}\right)^2 + ... + \left(1 + \frac{r}{12}\right)^{m-2} + \left(1 + \frac{r}{12}\right)^{m-1}\right]$$

But hey! The bit in the square brackets is a geometric series, the first term being 1, the ratio being $\left(1 + \frac{r}{12}\right)$ and the number of terms being m. There is a formula for adding up the terms of a geometric series (see

Appendix A) which we can use to add these terms up. So the sum of the series, S_m will be

$$\begin{aligned} S_m &= 1 + \left(1+\frac{r}{12}\right) + \left(1+\frac{r}{12}\right)^2 + \ldots + \left(1+\frac{r}{12}\right)^{m-2} + \left(1+\frac{r}{12}\right)^{m-1} \\ &= \frac{1 \times \left[\left(1+\frac{r}{12}\right)^m - 1\right]}{\left(1+\frac{r}{12}\right) - 1} \\ &= \frac{\left(1+\frac{r}{12}\right)^m - 1}{\frac{r}{12}} \end{aligned}$$

So after m months, our outstanding debt will be

$$D_m = A\left(1+\frac{r}{12}\right)^m - p\left[\frac{\left(1+\frac{r}{12}\right)^m - 1}{\frac{r}{12}}\right]$$

And again, if after m months we have paid off the mortgage, then

$$0 = A\left(1+\frac{r}{12}\right)^m - p\left[\frac{\left(1+\frac{r}{12}\right)^m - 1}{\frac{r}{12}}\right]$$

and so

$$A\left(1+\frac{r}{12}\right)^m = p\left[\frac{\left(1+\frac{r}{12}\right)^m - 1}{\frac{r}{12}}\right]$$

so that

$$p = \frac{rA\left(1+\frac{r}{12}\right)^m}{12\left[\left(1+\frac{r}{12}\right)^m - 1\right]}$$

Now if there are twelve months in a year, then $m = 12n$, so finally,

$$p = \frac{r\left(1+\frac{r}{12}\right)^{12n}}{12\left[\left(1+\frac{r}{12}\right)^{12n} - 1\right]} A \qquad (2)$$

4 Interest Added Daily

Now if interest is added *daily*, interest is added more often than payments. Let's see how this pans out.

The First Month

First of all, if interest is added daily, then the annual interest rate, r needs to be distributed evenly over the year. Let's assume for a moment that there are 360 days in a year, then the *daily interest rate* will be $\frac{r}{360}$.

So, if we borrow £A at the start of our mortgage, then there will be a number of days that we will accumulate interest before we make a payment. Again for simplicity, let's assume that we make payments monthly, and each month consists of 30 days (so that there are $30 \times 12 = 360$ days in the year!). Then interest will accrue for 30 days before we make a payment.

That means that at the end of the first month, the amount still left to repay will be

$$D_1 = A\left(1 + \frac{r}{360}\right)^{30} - p$$

The Second Month

The outstanding debt from the end of the first month is now subject to a month's worth of daily interest, followed by a reduction due to a monthly repayment. So at the end of the second month,

$$D_2 = \left[A\left(1 + \frac{r}{360}\right)^{30} - p\right]\left(1 + \frac{r}{360}\right)^{30} - p$$

If we multiply this out, we get

$$D_2 = A\left(1 + \frac{r}{360}\right)^{2\times 30} - p\left(1 + \frac{r}{360}\right)^{30} - p$$

The Third Month

The outstanding debt from the end of the second month is now subject to a month's worth of daily interest, followed by a reduction due to a monthly repayment. So at the end of the third month,

$$D_3 = \left[A\left(1 + \frac{r}{360}\right)^{2\times 30} - p\left(1 + \frac{r}{360}\right)^{30} - p\right]\left(1 + \frac{r}{360}\right)^{30} - p$$

If we multiply this out, we get

$$D_3 = A\left(1 + \frac{r}{360}\right)^{3\times 30} - p\left(1 + \frac{r}{360}\right)^{2\times 30} - p\left(1 + \frac{r}{360}\right)^{30} - p$$

The m^{th} Month

We're starting to see a pattern similar to the ones above!

The outstanding debt from the end of the m^{th} month will be

$$D_m = A\left(1 + \frac{r}{360}\right)^{m\times 30} - p\left(1 + \frac{r}{360}\right)^{(m-1)\times 30} - p\left(1 + \frac{r}{360}\right)^{(m-2)\times 30} ... - p\left(1 + \frac{r}{360}\right)^{30} - p$$

We can write this as

$$D_m = A\left(1 + \frac{r}{360}\right)^{m\times 30} - p\left[\left(1 + \frac{r}{360}\right)^{(m-1)\times 30} + \left(1 + \frac{r}{360}\right)^{(m-2)\times 30} ... + \left(1 + \frac{r}{360}\right)^{30} + 1\right]$$

The bit in the square brackets is again a geometric series: this time the first term is 1, the ratio is $\left(1+\frac{r}{360}\right)^{30}$ and the number of terms is m. Using the sum of a geometric series formula, the sum of this series will be

$$S_m = \frac{1 \times \left[\left\{\left(1+\frac{r}{360}\right)^{30}\right\}^m - 1\right]}{\left(1+\frac{r}{360}\right)^{30} - 1}$$

or

$$S_m = \frac{\left(1+\frac{r}{360}\right)^{30m} - 1}{\left(1+\frac{r}{360}\right)^{30} - 1}$$

So, the amount of outstanding debt in our mortgage after m months will be

$$D_m = A\left(1+\frac{r}{360}\right)^{30m} - p\frac{\left(1+\frac{r}{360}\right)^{30m} - 1}{\left(1+\frac{r}{360}\right)^{30} - 1}$$

But after m months, let's say that our mortgage has been repaid, so $D_m = 0$. In that case,

$$A\left(1+\frac{r}{360}\right)^{30m} = p\frac{\left(1+\frac{r}{360}\right)^{30m} - 1}{\left(1+\frac{r}{360}\right)^{30} - 1}$$

or

$$p = \frac{\left[\left(1+\frac{r}{360}\right)^{30} - 1\right] A\left(1+\frac{r}{360}\right)^{30m}}{\left(1+\frac{r}{360}\right)^{30m} - 1}$$

Now if there are twelve months in a year, then $m = 12n$, so finally,

$$p = \frac{\left[\left(1+\frac{r}{360}\right)^{30} - 1\right]\left(1+\frac{r}{360}\right)^{360n}}{\left(1+\frac{r}{360}\right)^{360n} - 1} A \tag{3}$$

5 Reality

But there aren't 360 days in a year! So a better approximation for our mortgage payments would surely be

$$p = \frac{\left[\left(1+\frac{r}{365.25}\right)^{30.4375} - 1\right]\left(1+\frac{r}{365.25}\right)^{365.25n}}{\left(1+\frac{r}{365.25}\right)^{365.25n} - 1} A \tag{4}$$

since *on average* there are about 365.25 days in a year (accounting for *leap years*), and $\frac{365.25}{12} = 30.4375$ days in a month.

This is the best model to use to calculate real mortgage payments. Banks and Building Societies compound interest daily. Equation (4) is the formula used by on-line mortgage payment calculators.

6 Examples

If you look at all the equations for the monthly payments, you will find that they are all essentially of the form

$$p = \text{factor} \times A$$

where the *factor* is a function of the interest rate and the length of the term. So what I'm going to do here is to show you calculations of p for different interest rates and terms, but all for a loan of £100,000.

If you want to find out your payment for a loan of £150,000, then you would just multiply the value by 1.5.

If you want to find out your payment for a loan of £200,000, then you would just double it. Get the idea?

So, Table 1 shows the monthly payments for a variety of interest rates, and for the two most common terms of 25 and 30 years for a loan of £100,000. If you want the monthly payment for a rate and/or a term not listed in the table, then you will have to use Equation (4).

What? I hear you cry. An interest rate of 15%? Never. Surely.

In November 1982, the National Average Contract Mortgage Rate[4] peaked at 15.80%. Scared? You should be.

[4] The National Average Contract Mortgage Rate (NACMR) is the average contract rate reported by a sample of mortgage lenders for fully amortized mortgage loans extended for the purchase of single family residences that were closed during the last 5 working days of the month. Before November 1991, the index was calculated based on the contract rates for loans closed during the first 5 working days of the month. Since November 1991, the index is calculated based on the contract rates for loans closed during the last 5 working days of the month.

Annual Interest Rate	25 Years	30 Years
1.00	£376.89	£321.66
1.50	£399.98	£345.16
2.00	£423.93	£369.70
2.50	£448.74	£395.25
3.00	£474.40	£421.80
3.50	£500.89	£449.32
4.00	£528.19	£477.79
4.50	£556.30	£507.17
5.00	£585.18	£537.44
5.50	£614.82	£568.56
6.00	£645.19	£600.48
6.50	£676.27	£633.19
7.00	£708.04	£666.63
7.50	£740.47	£700.77
8.00	£773.53	£735.57
8.50	£807.19	£770.98
9.00	£841.44	£806.98
9.50	£876.23	£843.52
10.00	£911.55	£880.56
10.50	£947.37	£918.07
11.00	£983.65	£956.02
11.50	£1020.38	£994.37
12.00	£1057.53	£1033.10
12.50	£1095.07	£1072.16
13.00	£1132.98	£1111.54
13.50	£1171.24	£1151.21
14.00	£1209.83	£1191.15
14.50	£1248.72	£1231.33
15.00	£1287.90	£1271.73

Table 1: Monthly Payments for a Loan of £100,000

7 Formula Summary

Interest Added...	Monthly Mortgage Payment
Yearly	$\frac{r(1+r)^n}{12[(1+r)^n-1]}A$
Monthly	$\frac{r\left(1+\frac{r}{12}\right)^{12n}}{12\left[\left(1+\frac{r}{12}\right)^{12n}-1\right]}A$
Daily	$\frac{\left[\left(1+\frac{r}{365.25}\right)^{30.4375}-1\right]\left(1+\frac{r}{365.25}\right)^{365.25n}}{\left(1+\frac{r}{365.25}\right)^{365.25n}-1}A$

Table 2: Monthly Payment Formulas

A Geometric Sequences and Series

A *geometric sequence* is a sequence of numbers that is formed by picking a starting number (usually known as a), and another number (known as the *ratio*, or r). Each *term* of the sequence is obtained by *multiplying* the previous term by r.

So, as an example, here is a geometric sequence, where $a = 2$ and $r = 3$:

$$2 \quad 6 \quad 18 \quad 54 \quad 162 \quad ...$$

And here is another geometric sequence, where $a = 2$ and $r = \frac{1}{3}$:

$$2 \quad \frac{2}{3} \quad \frac{2}{9} \quad \frac{2}{27} \quad \frac{2}{81} \quad ...$$

A.1 The Terms of a Geometric Sequence

In general, we can write any geometric sequence like this:

$$a \quad ar \quad ar^2 \quad ar^3 \quad ar^4 \quad ...$$

so that the 1^{st} term (known as U_1) is a, the 3^{rd} term (U_3) is ar^2, the 5^{th} term (U_5) is ar^4, etc. Have you spotted a pattern? Well there is one. The n^{th} term of a geometric series (known as U_n) is given by

$$U_n = ar^{n-1} \tag{5}$$

A.2 The Sum of n Terms of a Geometric Sequence

You might want to add up terms in a geometric sequence[5]. The sum of the first n terms of a geometric sequence (S_n) would be

$$S_n = a + ar + ar^2 + ar^3 + ... + ar^{n-2} + ar^{n-1} \tag{6}$$

There's a neat trick that you can use to work out how to add up all these terms. Multiply both sides of (6) by r:

$$rS_n = ar + ar^2 + ar^3 + ar^4 + ... + ar^{n-1} + ar^n \tag{7}$$

Now subtract (7) from (6)! Since most of the terms are the same, a lot of them cancel and you are left with

$$S_n - rS_n = a - ar^n$$

so that

$$S_n(1 - r) = a(1 - r^n)$$

and so

$$S_n = \frac{a(1 - r^n)}{1 - r} \tag{8}$$

Now you could have chosen to subtract (6) from (7) instead! In that case you would have ended up with

$$S_n = \frac{a(r^n - 1)}{r - 1} \tag{9}$$

So, which of the equations (8) or (9) do you use? Well, it's up to you. You can pick the one that's most convenient. For example, if $r > 1$ then (9) would be more convenient, as you not incur any nasty $-$ signs. And if $r < 1$ then (8) would be more convenient, for the same reason. But you can actually use either equation whatever problem you had.

[5] The sum of terms of a sequence is known as a *series*.

A.3 The Sum of an Infinite Number of Terms of a Geometric Sequence

Now a *really* cool thing turns up with geometric sequences. Check this out.

If $r < 1$, then $r^2 < r$, right? And $r^3 < r^2$. And $r^4 < r^3$, etc. Try it and see, if you're not convinced. And in fact, as n gets larger and larger, r^n gets smaller and smaller. So if we pick a really big n, r^n is essentially zero.

So what if n tended to *infinity*? Well then r^n would tend to 0.

So, using Equation (8), since $r < 1$, remember,

$$\begin{aligned} S_\infty &\to \frac{a(1-0)}{1-r} \\ &= \frac{a}{1-r} \end{aligned}$$

This is amazing! That would mean then that for our second example, where $a = 2$ and $r = \frac{1}{3}$,

$$2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \frac{2}{81} + \text{...(forever!!)}$$

would be equal to

$$\begin{aligned} S_\infty &= \frac{2}{1-\frac{1}{3}} \\ &= \frac{2}{\frac{2}{3}} \\ &= 3 \end{aligned}$$

So it's possible to add up *an infinite number of terms and get a finite result*!!

Don't you just love maths?