Steve Smith Tuition: Maths Notes

Function Transformations 2: Reflections and Modulus Operations

Contents

Prerequisites

You should read my document "Function Transformations 1: Shifts and Scalings" (see Smith (2013)) before reading this one!

Notes

This topic is covered by two documents: this one, and it's prequel, "Function Transformations 1: Shifts and Scalings" (see Smith (2013)).

Smith (2013) covers the basics of function transformations, and goes on to discuss shifts (left and right) and scalings (compressions and stretches) in both x and y directions.

This document covers the additional topics of reflections and modulus operations on functions (whatever they are).

Document History

Date	Version	Comments
22th October 2013	1.0	Initial creation of the document.

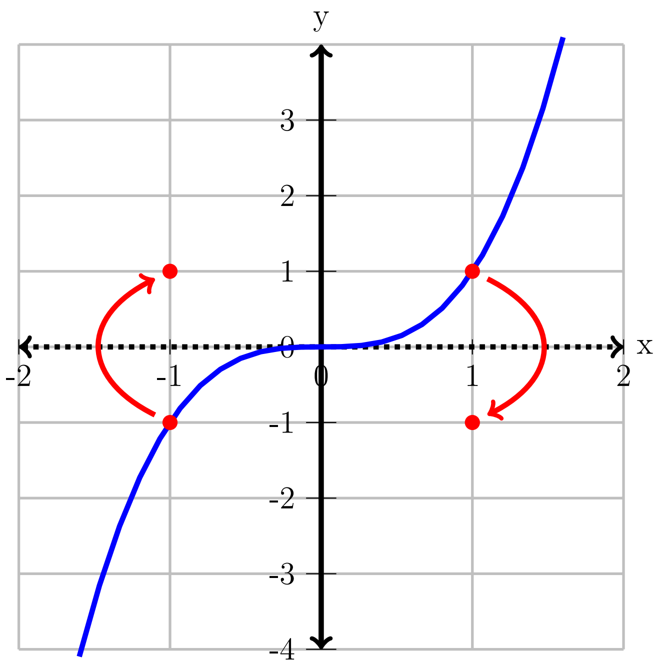
1 Reflections

1.1 Reflections in the x-Axis

The blue curve in Figure 1 is a graph of the function $y = x^3$. Let's think for a minute how we could achieve a reflection in the x-axis.

Now if we wanted to reflect this line in the x-axis, then, for example, the point $(1, 1)$ would be transformed to the point $(1, -1)$; and the point $(-1, -1)$ would be transformed to the point $(-1, 1)$ (see Figure 1).

Figure 1: A graph of $y = x^3$



Can you see how you would transform the point $(1, 1)$ into $(1, -1)$, and the point $(-1, -1)$ into $(-1, 1)$? In each case the x-coordinate is staying the same, but the sign of the y-coordinate is changing. Maybe that's it! Maybe the transformation is:

Figure 2: Is this the Transformation of a Reflection in the x-axis?

Well, let's try it. Figure 3 shows the function $y = x^3$ transformed by $y \to -y$.

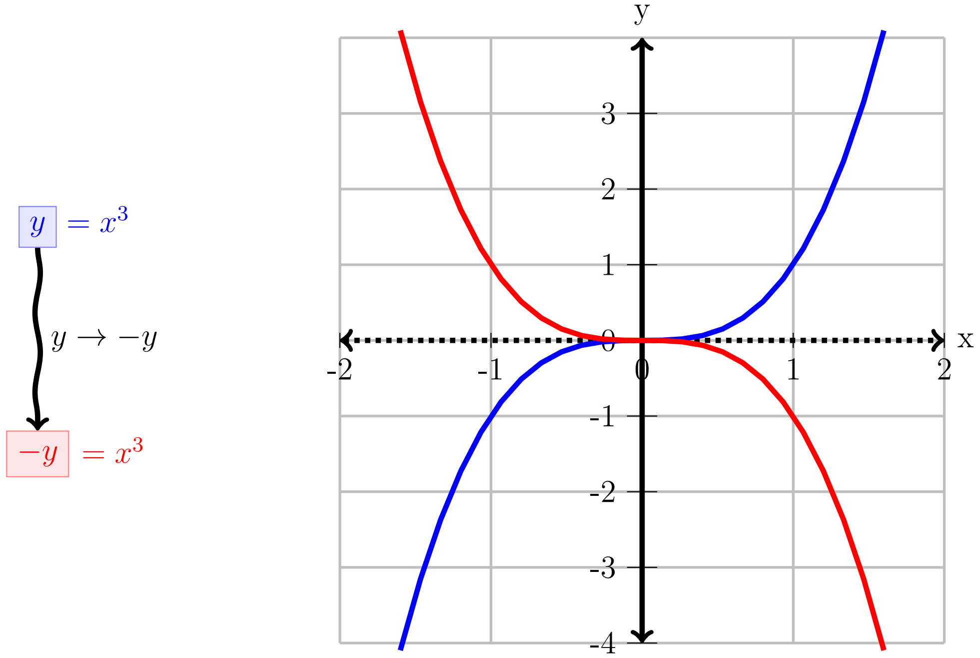
It seems to work! In this case the transformed equation will be

$$-y = x^3$$

which we can of course simplify to the usual format of

$$y = -x^3$$

Figure 3: Transforming $y = x^3$ by $y \to -y$

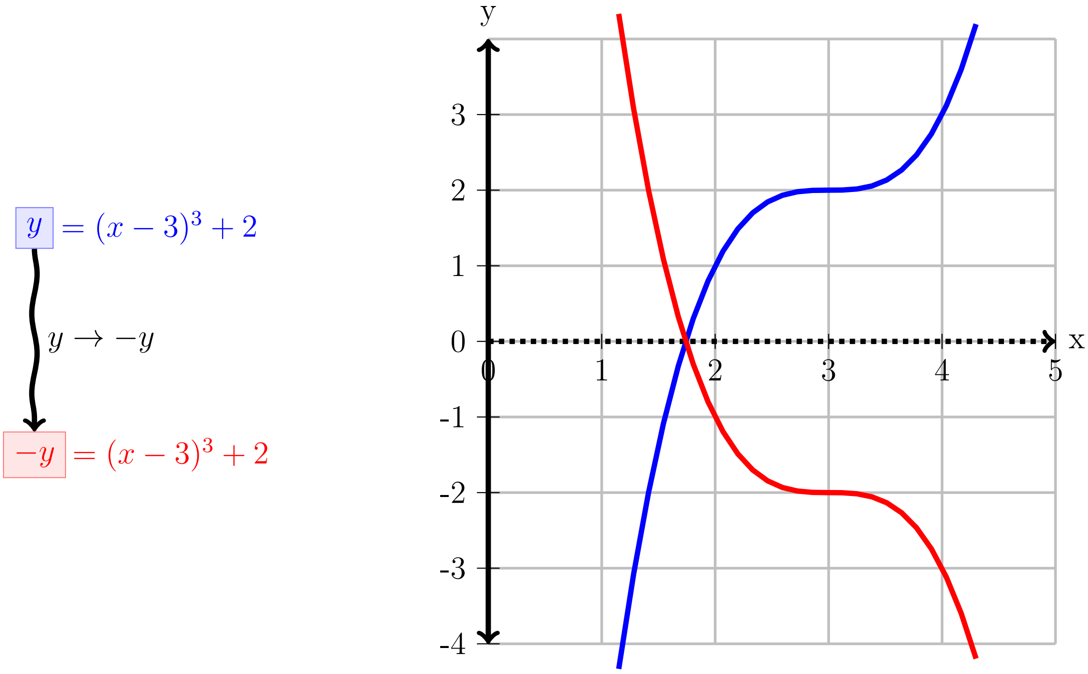


Let's try transforming another function this way, just so that we can be more confident we're on the right lines. Let's try transforming the equation

$$y = (x-3)^3 + 2$$

Figure 4 shows this transformation in action.

Figure 4: Transforming $y = (x-3)^3 + 2$ by $y \to -y$



This is looking good to me. Again, we would normally write the transformed equation

$$-y = (x-3)^3 + 2$$

as

$$y = -\left[(x-3)^3 + 2\right]$$

And it turns out that a reflection in the x-axis *is* given by the transformation $y \to -y$. Let's keep track of this discovery in a table. Check out Table 1.

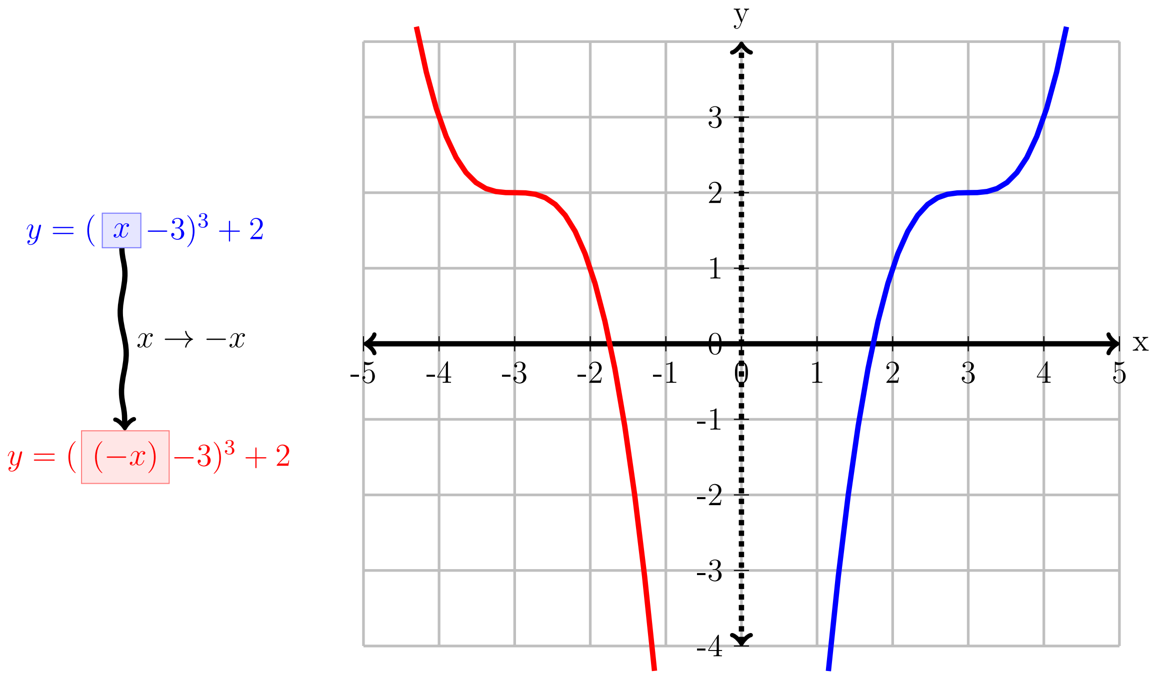
Table 1: Transformation Equivalents (1)

Equation Transformation	**Graph Transformation**
$y \to -y$	Reflection in the x-axis

1.2 Reflections in the y-Axis

Well I don't know about you, but I've had an idea! I was just thinking that if a reflection in the x-axis is given by the transformation $y \to -y$, then hey! Just maybe, a reflection in the y-axis might be given by the transformation $x \to -x$. What do you reckon? It does have a kind of symmetry about it, after all! Let's try this idea on the last function we used, $y = (x-3)^3 + 2$. Check out Figure 5:

Figure 5: Transforming $y = (x-3)^3 + 2$ by $x \to -x$



Well, shiver me timbers! It does work. Yippee!!

1.3 Summary of Reflection Transformations

Table 2 summarises reflection transformations.

Table 2: Transformation Equivalents (2)

Equation Transformation	**Graph Transformation**
$y \to -y$	Reflection in the x-axis
$x \to -x$	Reflection in the y-axis

2 Modulus Operations

To start with, what do I mean by a "modulus" operation? Well, in mathematics there is a modulus operation[1]. The modulus of a number, 5 say, is denoted by $|5|$. And it means "make me positive". That means that $|5| = 5$ because 5 is positive already. But if you take the modulus of a negative number, then for example, $|-3| = 3$.

2.1 Modulus of a Function

But you can also use modulus with functions, like this:

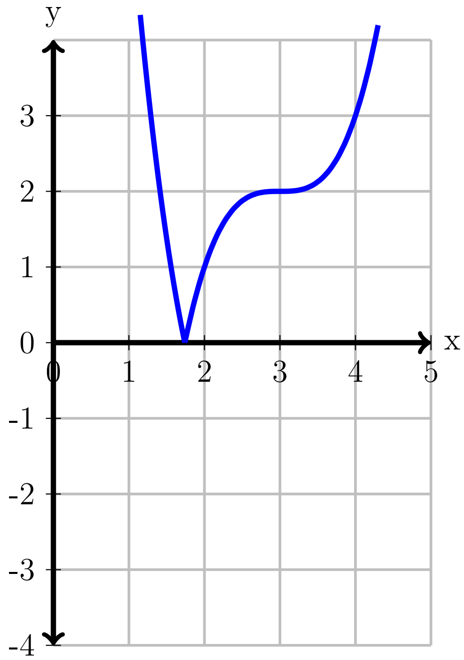
$$y = \left|(x-3)^3 + 2\right|$$

Now what does this mean? Well, in order to work out the value that y would be for any given value of x, what you need to do is to just evaluate the function for that value of x, then "make it positive". For example, with the function $y = |(x-2)^3 + 2|$, evaluated at $x = 1$, here's what we would get:

$$\begin{aligned} y(1) &= \left|(1-3)^3 + 2\right| \\ &= \left|(-2)^3 + 2\right| \\ &= |-8 + 2| \\ &= |-6| \\ &= 6 \end{aligned}$$

OK, so what would a graph of $y = |(x-3)^3 + 2|$ look like? Well, if you go through the process of constructing a table of values for x and y as I discussed in the early stages of Smith (2013), then plot the points, you get something like this: Compare this with the blue line in Figure 5. That's the

Figure 6: A graph of $y = |(x-3)^3 + 2|$



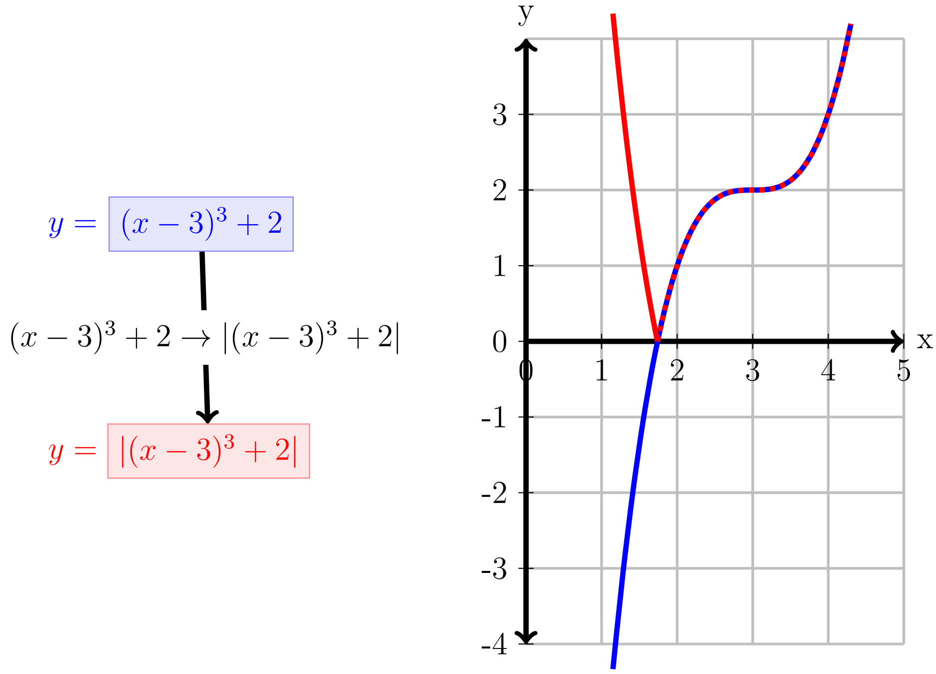
ordinary function $y = (x-3)^3 + 2$ with no modulus. Can you see what's happened? Points that

[1] I call it an *operation* because it's not a function. More on this later...

were above the x-axis before (i.e. *positive* values of y) will be unaffected by the modulus, so they stay where they were. But points that were below the x-axis before (i.e. *negative* values of y) will be affected by the modulus: they will be made positive. So the bit below the x-axis before will be *reflected* above it!

So we can think of this as a transformation[2]. See Figure 7.

Figure 7: The Modulus Transformation of $y = (x-3)^3 + 2$ by $(x-3)^3 + 2 \to |(x-3)^3 + 2|$



Here, the transformed function, $y = |(x-3)^3 + 2|$ overlaps the original blue function, $y = (x-3)^3+2$, when the y value of the orginal is positive. I've shown that in Figure 7 by the dotted blue/red line.

However, for negative y values of the original function, the graph is reflected up in the x-axis. Weird!

[2] I don't want to go into *why* the modulus operation not a function here, or the general consequences of this, but it does mean that in this particular case of taking the modulus of a function, we have to break our neat symmetry rules for x and y that we discussed at length in Smith (2013). What we *want* to do at this point is say that taking the modulus of a function would be equivalent to the transformation $y \to |y|$. But we can't. We have to say instead that if you have the function $y = f(x)$, and you want to perform the modulus transformation, then $y = f(x)$ will be transformed to $y = |f(x)|$. Don't blame me!

2.2 Modulus of the x Variable

The other thing we can do with the modulus operation is something like this. If we started with the function

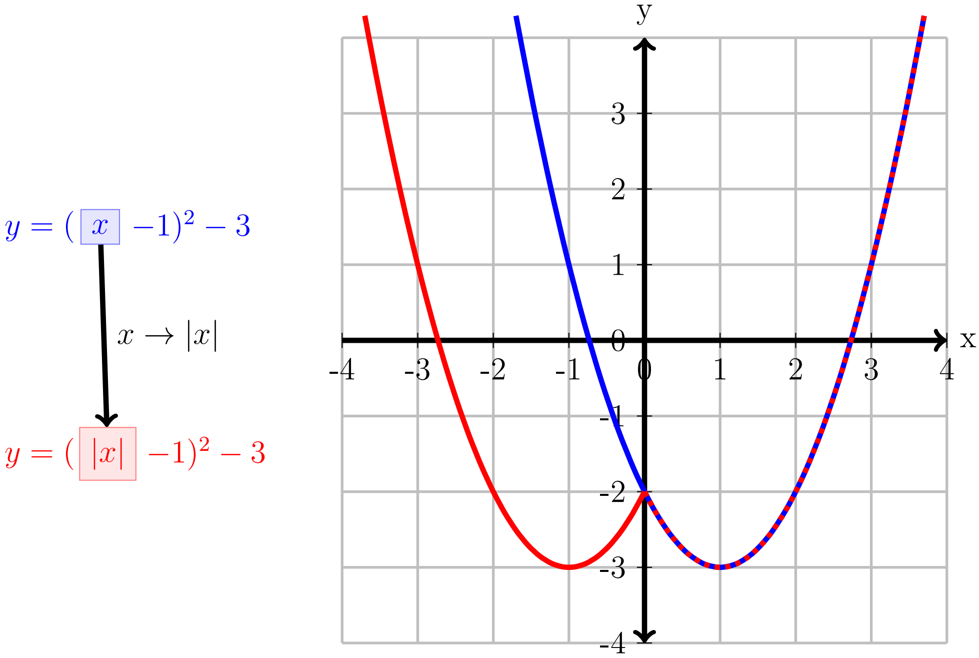
$$y = (x-1)^2 - 3$$

say, then we could always perform the transformation $x \to |x|$, so that our function became

$$y = (|x|-1)^2 - 3$$

OK, so what would a graph of this transformation look like? Have a look at Figure 8 to see for yourself. Now let's see if we can make some sense of this. The original equation, $y = (x-1)^2 - 3$ is given by

Figure 8: The Transformation of $y = (x-1)^2 - 3$ by $x \to |x|$



the blue line. I'm happy with that - it's just a normal quadratic. But the red line, corresponding to $y = (|x|-1)^2 - 3$, looks a bit weird. Part of it is in the same place as the blue line (shown in Figure 8) by the blue/red dotted bit), but part of it isn't.

Let's start trying to understand this by looking at what happens when x is positive, and then when x is negative[3].

So, when x is positive, then taking the modulus of x will have absolutely no effect, since x is positive already. That means that $y = (x-1)^2 - 3$ and $y = (|x|-1)^2 - 3$ will be exactly the same! Ah - that means that for positive x, the two graphs will overlap. And that's exactly what we see in Figure 8!

Right, now when x is negative, then the modulus *will* make a difference. Let's say that we picked a value of x of -1, and we wanted to find the y value corresponding to this value of x. This is what would happen:

[3]I've done this because it's the sign of x that the modulus operation is going to change!

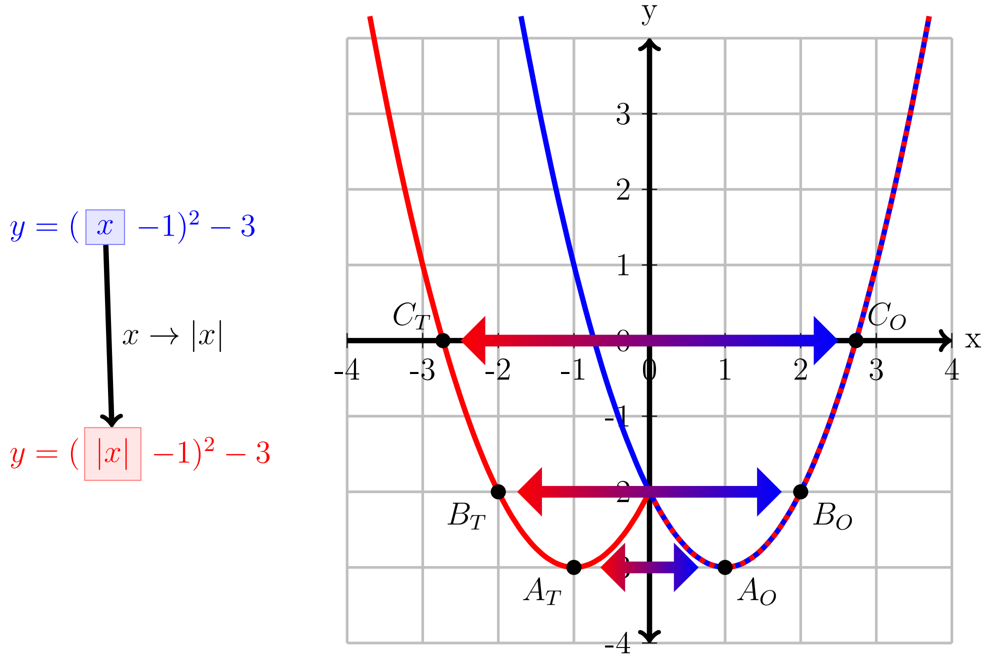
- First, we have to do the modulus thing, i.e. make it positive, so $|x|$ would be 1;
- Then we evaluate the function: $y = (|x| - 1)^2 - 3 = (1 - 1)^2 - 3 = 0^2 - 3 = -3$;
- And so the y value corresponding to $x = -1$ is -3 so finally we plot the point $(-1, -3)$.

So whenever x is negative, we essentially plot the point:

$$(x, \textit{the } y \textit{ value of the positive value of } x)$$

Have a look at Figure 9.

Figure 9: The Transformation of $y = (x - 1)^2 - 3$ by $x \to |x|$ Reprise



For example, to plot the point A_T, we take the x value (-1), make it positive (1), then find the y value of the point with $x = 1$. Well, the point with $x = 1$ is A_O, and the y value of A_O is -3. So A_T is at $(-1, -3)$.

And to plot the point B_T, we take the x value (-2), make it positive (2), then find the y value of the point with $x = 2$. Well, the point with $x = 2$ is B_O, and the y value of B_O is -2. So B_T is at $(-2, -2)$.

And to plot the point C_T, we take the x value $(-2.732$, to 3 decimal places), make it positive (2.732), then find the y value of the point with $x = 2.732$. Well, the point with $x = 2.732$ is C_O, and the y value of C_O is 0. So C_T is at $(-2.732, 0)$.

Now can you see what's happened? For negative x, we're just using the graph on the other side of the y axis. We've *reflected* the part of the graph where $x > 0$ in the y axis.

2.3 Summary of Modulus Transformations

Table 3: Modulus Transformation Equivalents

Equation Transformation	**Graph Transformation**
$y = f(x) \to y = \lvert f(x) \rvert$	$f(x) > 0 :$ No change $f(x) < 0 :$ Reflection in the x-axis
$x \to \lvert x \rvert$	$x > 0 :$ No change $x < 0 :$ Reflection in the y-axis

References

Smith, S. (2013). Function Transformations I. Explains how equations and graphs are transformed with shifts and scalings.