Steve Smith Tuition: Maths Notes

Function Transformations 1: Shifts and Scalings

Contents

Prerequisites

None.

Notes

This topic is covered by two documents: this one, and it's sequel, "Function Transformations 2: Reflections and Modulus Operations" (see Smith (2013)).

This document covers the basics of function transformations, and goes on to discuss shifts (left and right) and scalings (compressions and stretches) in both x and y directions.

Smith (2013) covers the additional topics of reflections and modulus operations on functions (whatever they are).

Document History

Date	Version	Comments
22th October 2013	1.0	Initial creation of the document.

1 Introduction

1.1 Sketching a Graph of a Function

There's going to be a lot of drawing of graphs in the next few pages, so let's go over how to sketch a graph of a function. Let's say that you want to draw a graph of the function

$$y = x^2$$

but you're not sure how to do it.

Well, one sure way of drawing a graph of the function $y = x^2$ is make a table like Table 1, where you pick some values of x, figure out what the corresponding y values will be (remember that $y = x^2$), plot the points, and joint the dots.

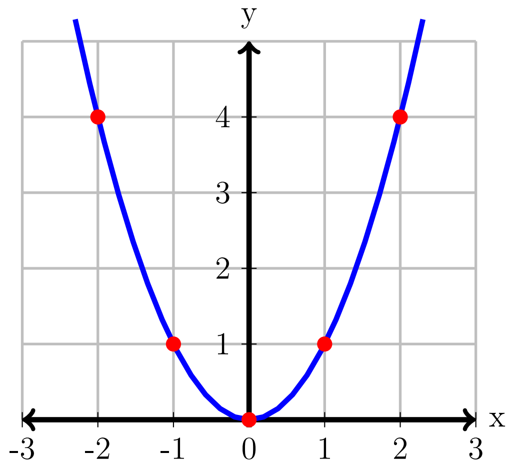
Table 1: Table of Values for $y = x^2$

x	-2	-1	0	1	2
y	4	1	0	1	4

If you do that, you'll end up with something that looks a bit like Figure 1. And if you wanted to

Figure 1: A graph of $y = x^2$

$y = x^2$



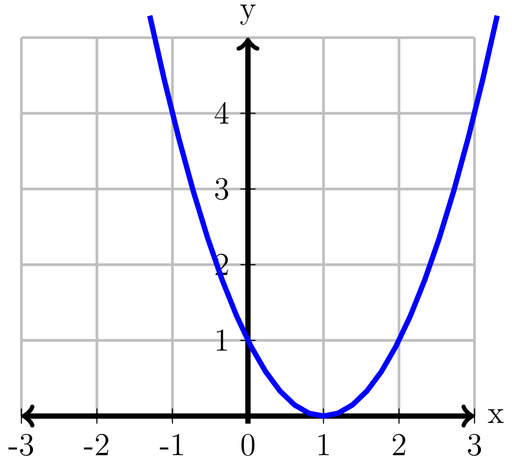
draw a graph of any other function, you'd do the same kind of thing: construct a table of x values, figure out the corresponding y values, plot the points, and joint the dots.

1.2 Transforming a Graph

OK, so what's all this "transforming graph" stuff about? Well the idea is this. If you started with the function $y = x^2$, and you sketched the graph of it (remember Figure 1), then what would the graph look like if you shifted the function 1 unit to the right? Well, that's not really hard to figure out: it's going to look like the graph in Figure 2. Now what's not quite so obvious is: what would be the equation of the function in Figure 2? How would we go about finding out?

And of course, shifting a graph 1 unit to the right isn't the only kind of "transformation" you could do to a graph. You could stretch it out, reflect it in some line, or do even more weird kinds of thing. And

Figure 2: A graph of $y =$???



all the time, each kind of transformation of the graph (of a function) would have its own equivalent transformation of the equation (of the function). The two go together. Whenever we transform a function, we will be transforming not only the graph of the function, but the equation of the function as well.

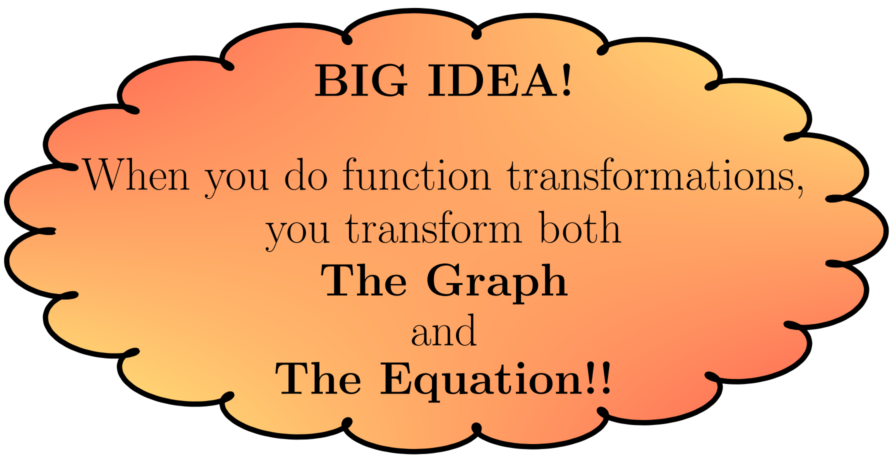


Figure 3: The Big Idea in Function Transformations!

To show you what I mean, have a look at what happens to the equation of the function $y = x^2$ when you shift the graph 1 unit to the right. The result is shown in Figure 4.[1]

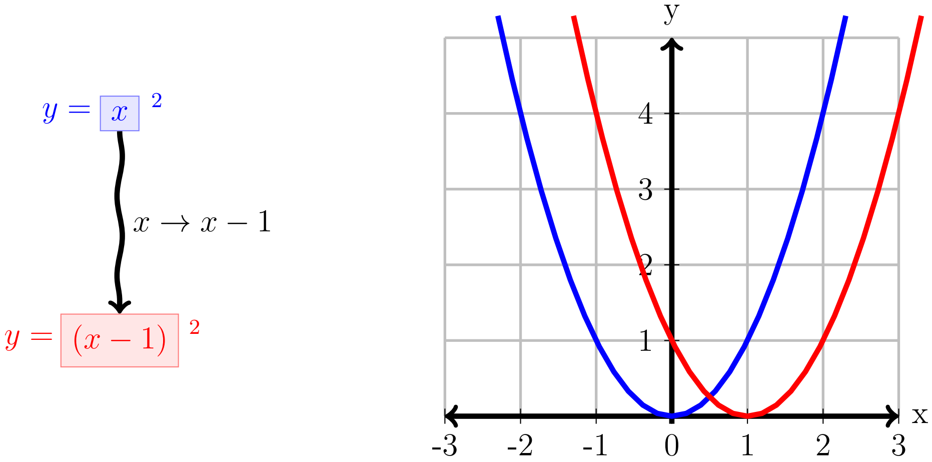
As you can see, the equation of the function has been transformed. Where we started with x in the original equation, we've ended up with $x - 1$ in the transformed equation. So there has been a transformation of $x \to x - 1$ in the equation when we shift the graph 1 unit to the right (i.e. 1 unit in the positive x-direction). So the equation of the transformed graph is

$$y = (x - 1)^2$$

We will see shortly that there are a multitude of different ways that you can transform a function, and each function transformation has it's own associated graph transformation and it's own associated equation transformation. So really this whole topic boils down to two important questions.

[1] In this figure, as in all following figures, the **blue** graph is the original function (in this case the $y = x^2$), and the **red** graph is the transformed function.

Figure 4: Transforming $y = x^2$ by shifting 1 unit to the right



1.3 The Two Important Questions

While studying function transformations, always keep in mind the following two questions. The whole function transformation business can be summed up by:

1. Given a transformation of a graph, what's the associated transformation of the equation?
2. Given a transformation of an equation, what's the associated transformation of the graph?

2 Types of Transformations

There are four types of transformation:

- Translations. Also known as "shifts". These can be performed in either the x (left or right) or y (up or down) directions;
- Transformations of scale. Also known as "stretches", or "compressions". These can be performed in either the x (stretching out or compressing in) or y (stretching out or compressing in) directions;
- Reflections. These can be reflections in the x-axis, or the y-axis (these are fully described in (Smith, 2013));
- Modulus transformations. These are a bit odd, and are fully described in (Smith, 2013).

3 Performing Transformations

3.1 Translation (or shift) Transformations

3.1.1 x-shifts

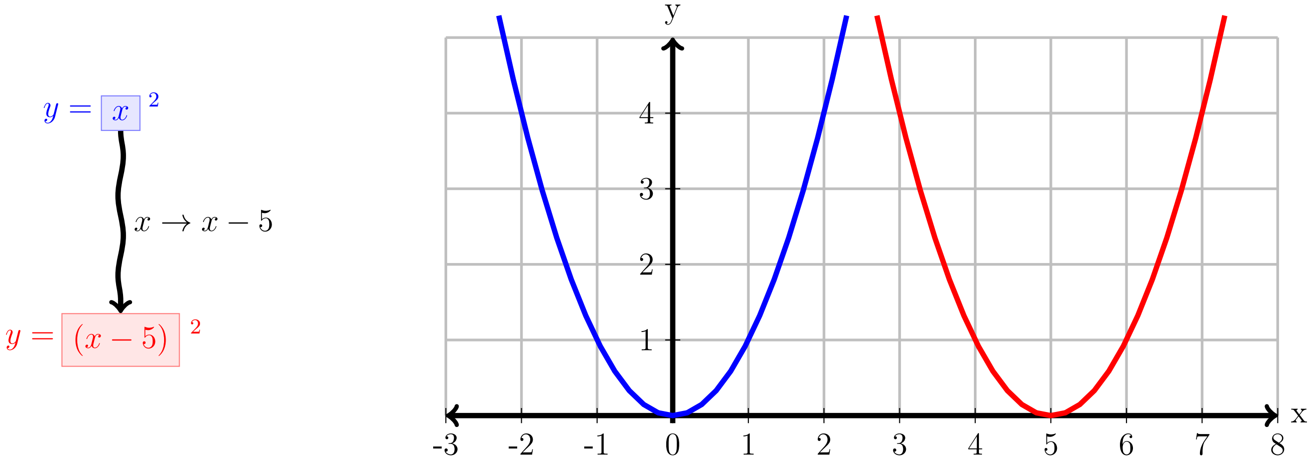
We've already seen an example of one of these. In section 1.2 we saw that a transformation of the equation of $x \to x - 1$ is equivalent to a shift of the graph 1 unit to the right. Let's put that in a table (Table 2), so that we can keep track of it. I'm going to add stuff to this table the more transformations we encounter.

Table 2: Transformation Equivalents (1)

Equation Transformation	**Graph Transformation**
$x \to x - 1$	x-shift by 1 to the right

Alright, so if your life was on the line, what do you think a transformation of a graph of 5 units to the right would be? Have a think about this before looking at Figure 5

Figure 5: Transforming $y = x^2$ by shifting 5 units to the right



Well, yes, OK, no great surprise here. A transformation (in the x-direction) 5 units to the right is equivalent to the equation transformation of $x \to x - 5$. Let's put that in our table, so we don't forget:

Table 3: Transformation Equivalents (2)

Equation Transformation	**Graph Transformation**
$x \to x - 1$	x-shift by 1 to the right
$x \to x - 5$	x-shift by 5 to the right

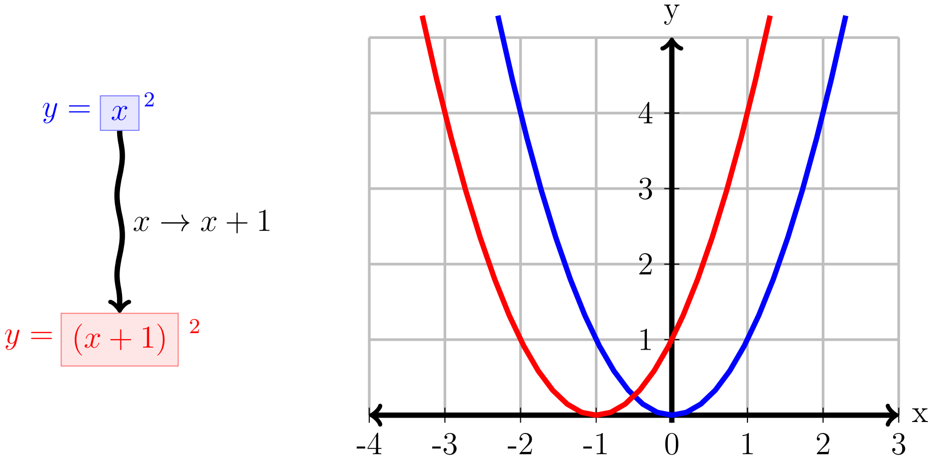
And you know what, if you were to bet that a transformation (in the x-direction) 37 units to the right is equivalent to the equation transformation of $x \to x - 37$, you'd be dead right. As mathematicians, we have a neat way of summarising this idea. It's called *algebra*. See Table 4.

Table 4: Transformation Equivalents (3)

Equation Transformation	**Graph Transformation**
$x \to x - a$	x-shift by a to the right

So now we've got a complete handle on shifting a graph by however much we like *to the right*. What about shifting a graph to the *left*? It turns out that shifting a graph 1 unit to the left is equivalent to an equation transformation of $x \to x + 1$. (If you had already guessed this, well done!) Let's have a look at the graph of this, when we transform $y = x^2$:

Figure 6: Transforming $y = x^2$ by shifting 1 unit1 to the left



And if we shift the graph 37 units to the left, that would be equivalent to an equation transformation of $x \to x + 37$, etc, etc. We can update our table again. Check out Table 5.

Table 5: Transformation Equivalents (4)

Equation Transformation	**Graph Transformation**
$x \to x - a$	x-shift by a to the right
$x \to x + a$	x-shift by a to the left

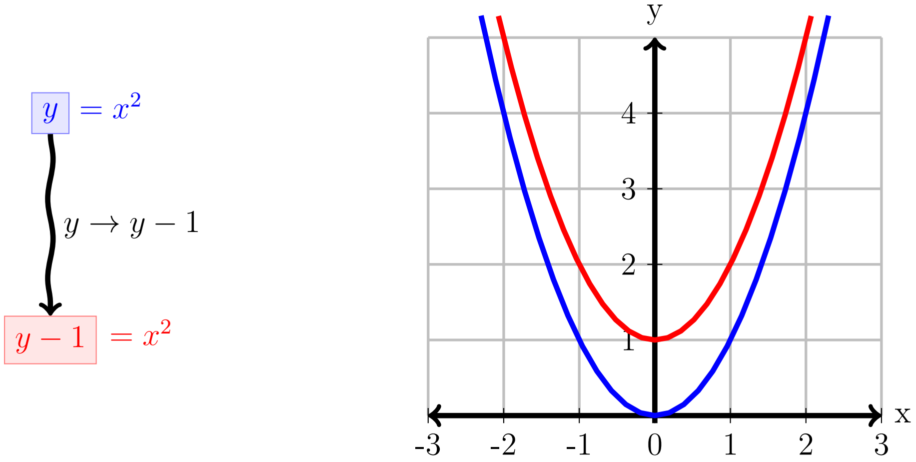
Let's just have a look at Table 5 for a minute. It seems a bit counter-intuitive to me. I mean, if we are shifting a graph to the *right*, I would perhaps have expected that since we are moving the graph toward more *positive* values of x then that might have meant that the equation would have been transformed by $x \to x + a$, but as you can see, that's not the case. In fact, with most of the transformations we encounter, this counter-intuitiveness seems to prevail. $x \to x + a$ means a shift to the *left* by a; $x \to x - a$ means a shift to the *right* by a.

3.1.2 y-shifts

Now then. Instead of thinking about x-shifts, let's think about shifting in the y-direction instead. What would happen if we shifted the graph of $y = x^2$ *up* by 1?

Now shifting a graph *up* is (1) shifting the graph in the y-direction, and (2) shifting toward more positive values of y. Thinking about how we dealt with x-shifts, does that give you a clue as to what might happen here? Take a look at Figure 7 In an exactly analogous way to what happened when

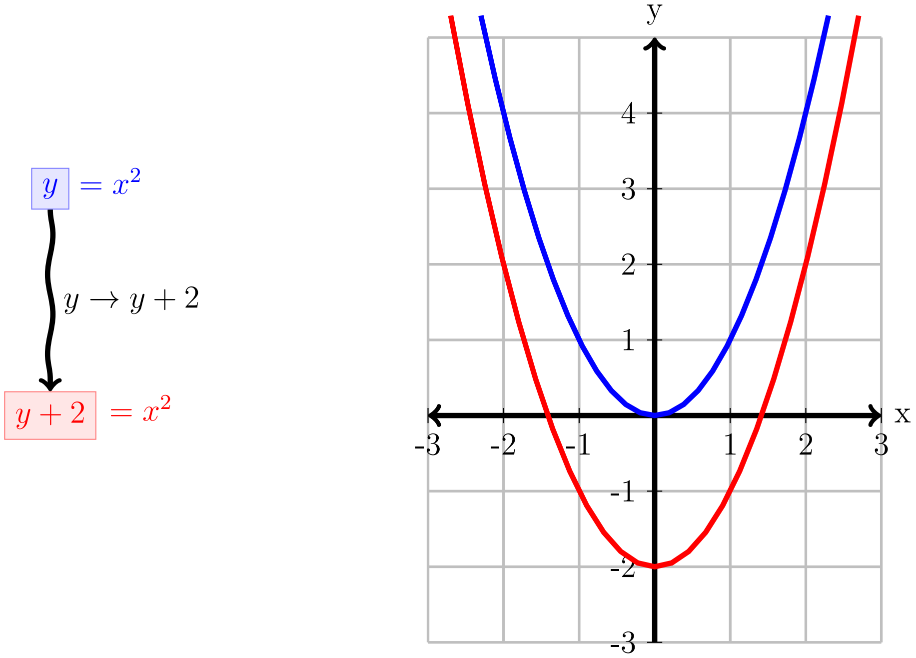
Figure 7: Transforming $y = x^2$ by shifting 1 unit up



we shifted in the x-direction toward more positive values of x ($x \to x - 1$), when we shift in the y-direction toward more positive values of y we get the eqution transformation $y \to y - 1$. The same thing happens to the y that happened to the x. Ah! consistency! It's a wonderful thing[2].

Now just so that we completely understand this shifting stuff, let's do a shift *down*. And let's shift down by 2. What do you think the equation will be transformed to? Have a look at Figure 8 to see if you were right.

Figure 8: Transforming $y = x^2$ by shifting 2 units down



We need to update our table:

[2]Talking of consistency, I'd like at this point to share with you my confusions about function transformations that I had when I was at school. As it's not really part of the story flow, I've put it in Appendix A.

Table 6: Transformation Equivalents (5)

Equation Transformation	**Graph Transformation**
$x \to x - a$	x-shift by a to the right
$x \to x + a$	x-shift by a to the left
$y \to y - a$	y-shift by a up
$y \to y + a$	y-shift by a down

3.1.3 Shift Transformation Summary

- Treat x- or y-shifts in a consistent way (do the same for either x or y);
- Shifts are achieved by *adding* or *subtracting* numbers from the x- or y-symbols in the equation;
- For x-shifts: adding is a *left*-shift, and subtracting is a *right*-shift (counter-intuitive);
- For y-shifts: adding is a *down*-shift, and subtracting is an *up*-shift (counter-intuitive).

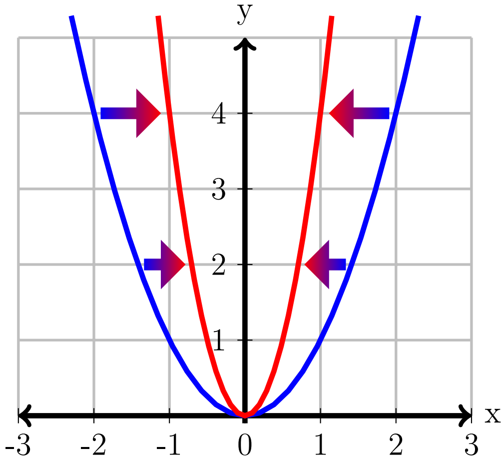
3.2 Scale Transformations

The next kind of transformation we're going to look at is called a *scaling*. What's one of those? Well a scaling is a stretch or a compression of a graph, in either the x- or y-directions.

3.2.1 x-Compressions

As a first example, let's have a look at a scaling of our old friend, $y = x^2$. This can be seen in Figure 9 It's kind of difficult to tell what's happened during this transformation, but take it from me that the

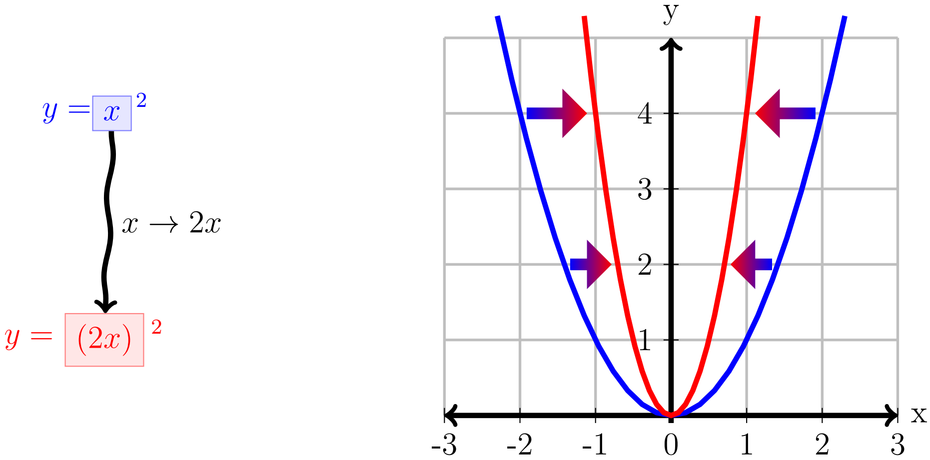
Figure 9: A graph of $y =$???



blue graph has been squeezed inwards (compressed), with what I call *a scale factor of 2*. That means that the x-values are all compressed by a factor of 2 (in other words half what they were before). The arrows on Figure 9 help you to see what's going on.

So the next question to ask would be: "What's the transformation of the equation that's equivalent to an x-compression, scale factor 2?". Well I'm glad you asked. Check out Figure 10 It turns out that

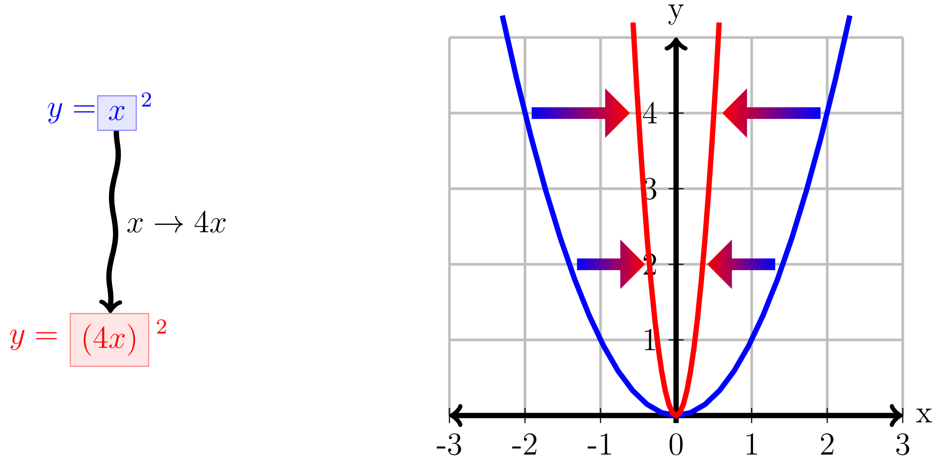
Figure 10: A graph of $y = (2x)^2$



the equation transformation that's equivalent to an x-compression, scale factor 2 is

$$x \to 2x$$

Figure 11: A graph of $y = (4x)^2$



OK, let's have another example. What about an x-compression, scale factor 4? See if you can guess what this graph would look like, before having a look at Figure 11. This time, the blue graph has been squeezed inwards (compressed), with a scale factor of 4. That means that the x-values are all compressed by a factor of 4 (in other words a quarter what they were before). The arrows on Figure 11 help you to see what's going on.

And the transformation of the equation that's equivalent to an x-compression, scale factor 4 is

$$x \to 4x$$

Hang on - I'm spotting a pattern here. I'll bet that an x-compression, scale factor a would be

$$x \to ax$$

and indeed it is. Let's update our table:

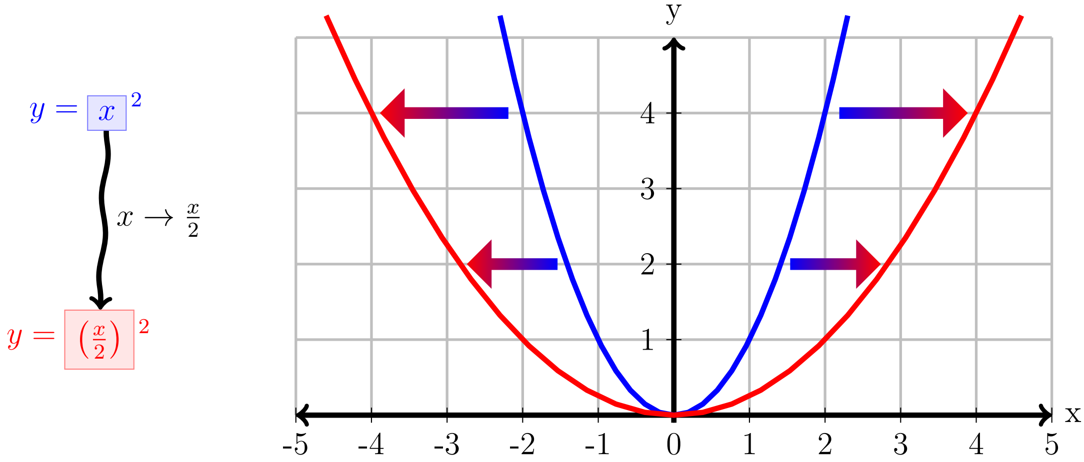
Table 7: Transformation Equivalents (6)

Equation Transformation	**Graph Transformation**
$x \to x - a$	x-shift by a to the right
$x \to x + a$	x-shift by a to the left
$y \to y - a$	y-shift by a up
$y \to y + a$	y-shift by a down
$x \to ax$	x-compression, scale factor a

3.2.2 x-Stretches

Alright. What about x-stretches? Let's say we wanted to stretch the graph of $y = x^2$ by a factor of 2 in the x-direction. Firstly, what would that look like? And secondly, what would be the equivalent equation transformation, do you think? The answers to both these exciting questions are given in Figure 12 It turns out that the transformation of the equation that's equivalent to an x-stretch, scale

Figure 12: A graph of $y = \left(\frac{x}{2}\right)^2$



factor 2 is

$$x \to \frac{x}{2}$$

It's interesting that just like in the case of x-shifts, what you do when compressing or stretching is kind of counter-intuitive. What I mean is that to compress a graph in the x-direction, you *multiply* the x in the equation by a number; and to stretch a graph in the x-direction, you *divide* the x in the equation by a number.[3]

Here's the new table:

Table 8: Transformation Equivalents (7)

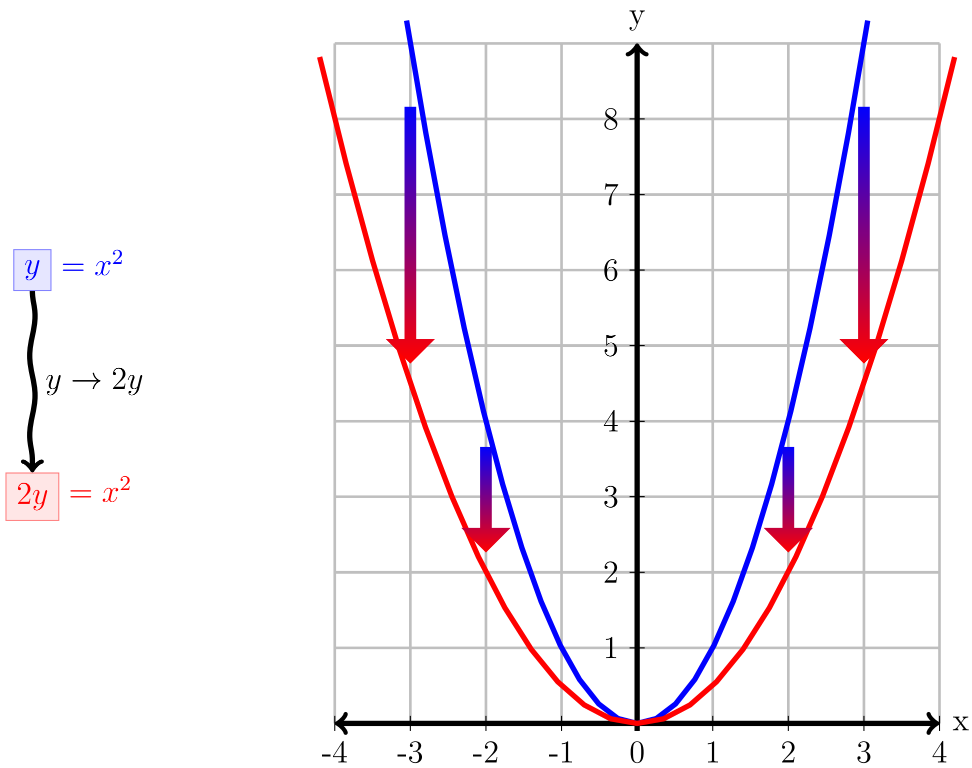
Equation Transformation	**Graph Transformation**
$x \to x - a$	x-shift by a to the right
$x \to x + a$	x-shift by a to the left
$y \to y - a$	y-shift by a up
$y \to y + a$	y-shift by a down
$x \to ax$	x-compression, scale factor a
$x \to \frac{x}{a}$	x-stretch, scale factor a

[3] It's not quite as simple as this: it matters how big your multiplying or dividing numbers are. Let's for now just assume that when we are multiplying or dividing one of our coordinates by a number, the number is bigger than 1. I'll come back to this issue in section B.

3.2.3 y-Compressions and y-Stretches

And guess what? What's good for the x-coordinate is good for the y-coordinate! As an example, let's take a y-compression, scale factor 2. See Figure 13 Just like with the case of an x-compression, the

Figure 13: A graph of $2y = x^2$



transformation in the equation for a y-compression, scale factor 2, is

$$y \to 2y$$

And if you wanted to do a y-stretch, scale factor 7, the equation transformation would be

$$y \to \frac{y}{7}$$

in exactly the same way that equations are transformed when doing x-stretches.

Once again we update our table, since we now know all there is to know about the possible shifts and scalings possible. I'm now going to split our table into two because it's getting a bit unweildly. I'm splitting it into separate tables, one for x transformations and one for y transformations. See how similar tables 9 and 10 are.

Hopefully this emphasises that whenever we do transformations, we should be transforming the x and the y in the equation by the equivalent kind of thing when we do equivalent x and y transformations of the graph. It's all completely consistent.

Table 9: All x Shift and Scaling Transformations

Equation Transformation	**Graph Transformation**
$x \to x - a$	x-shift by a to the right
$x \to x + a$	x-shift by a to the left
$x \to ax$	x-compression, scale factor a
$x \to \frac{x}{a}$	x-stretch, scale factor a

Table 10: All y Shift and Scaling Transformations

Equation Transformation	**Graph Transformation**
$y \to y - a$	y-shift by a up
$y \to y + a$	y-shift by a down
$y \to ay$	y-compression, scale factor a
$y \to \frac{y}{a}$	y-stretch, scale factor a

3.2.4 Scaling Transformation Summary

- Treat x- or y-scalings in a consistent way (do the same for either x or y);
- Scalings are achieved by *multiplying* or *dividing* the x- or y-symbols in the equation by numbers;
- For x-scalings: multiplying is a *compression*, and dividing is a *stretch* (counter-intuitive);
- For y-scalings: multiplying is a *compression*, and dividing is a *stretch* (counter-intuitive).

4 Transforming Actual Functions 1: from Graph to Equation

Right. That's the theory. Let's put it into practice. In these examples, we're not going to be transforming $y = x^2$ any more. We're onto more serious functions! And in this section, I'm going to be looking at the problem of trying to figure out what the transformed equation will be when you know what the transformation of the graph is.

4.1 Example 1 : Shifting $y = x^3 - 6x^2 + 9x$ to the Right

So here's our first example. Let's say that we wanted to transform the graph of:

$$y = x^3 - 6x^2 + 9x \tag{1}$$

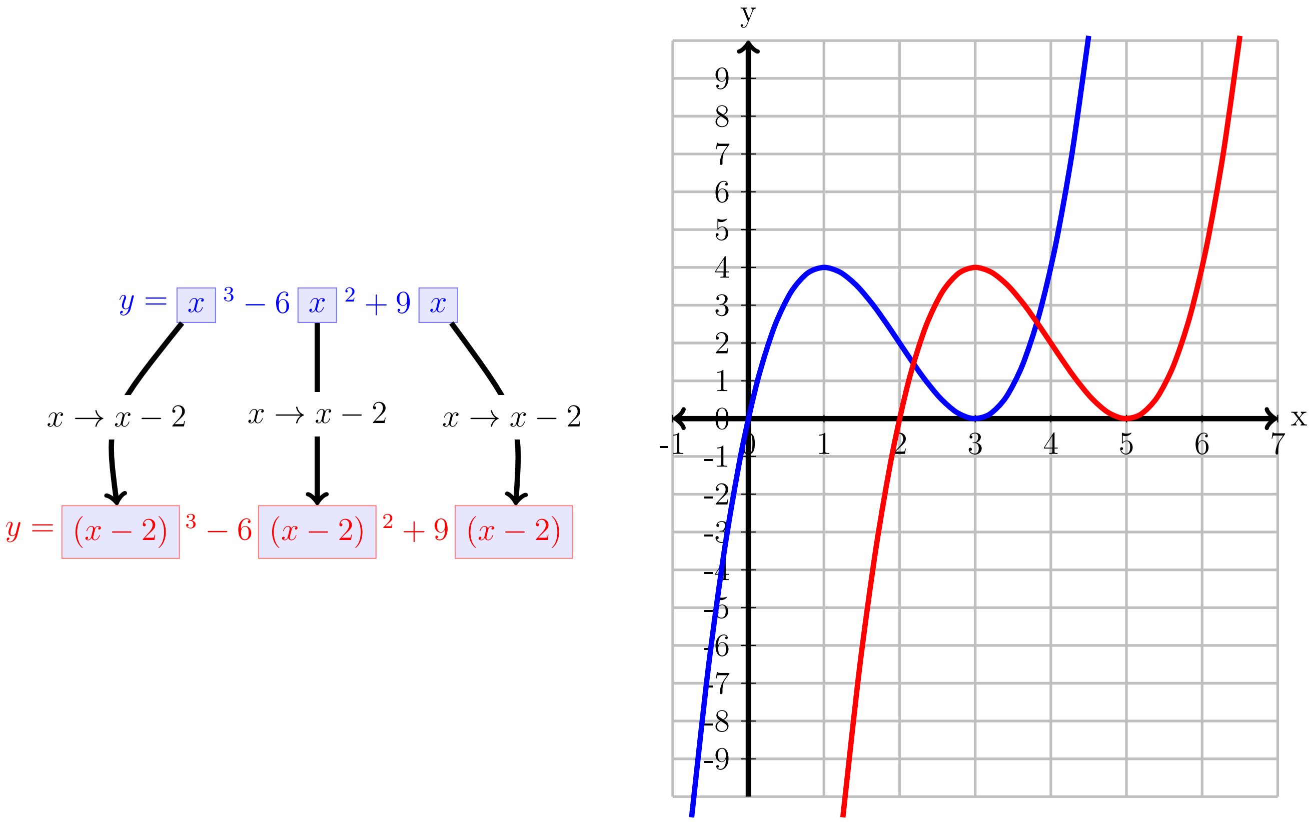
by shifting it 2 units to the right.

Do you remember what equation transformation was equivalent to an x-shift to the right by 2? It's:

$$x \to x - 2 \tag{2}$$

Now this time, we have more than one x in equation (1). So what do we do? We transform *every* x in equation (1) by the transformation (9): Have a look at Figure 14 for the solution:

Figure 14: Shifting $y = x^3 - 6x^2 + 9x$ Two Units to the Right



So the transformed equation will be:

$$y = (x - 2)^3 - 6(x - 2)^2 + 9(x - 2)$$

(which you could simplify by multiplying out the brackets if you really want to. I'm not going to bother!).

4.2 Example 2 : Scaling $y = x^3 - 6x^2 + 9x$ by Compressing in x, Scale Factor 3

So here's our second example. This time we want to transform the graph of:

$$y = x^3 - 6x^2 + 9x \tag{3}$$

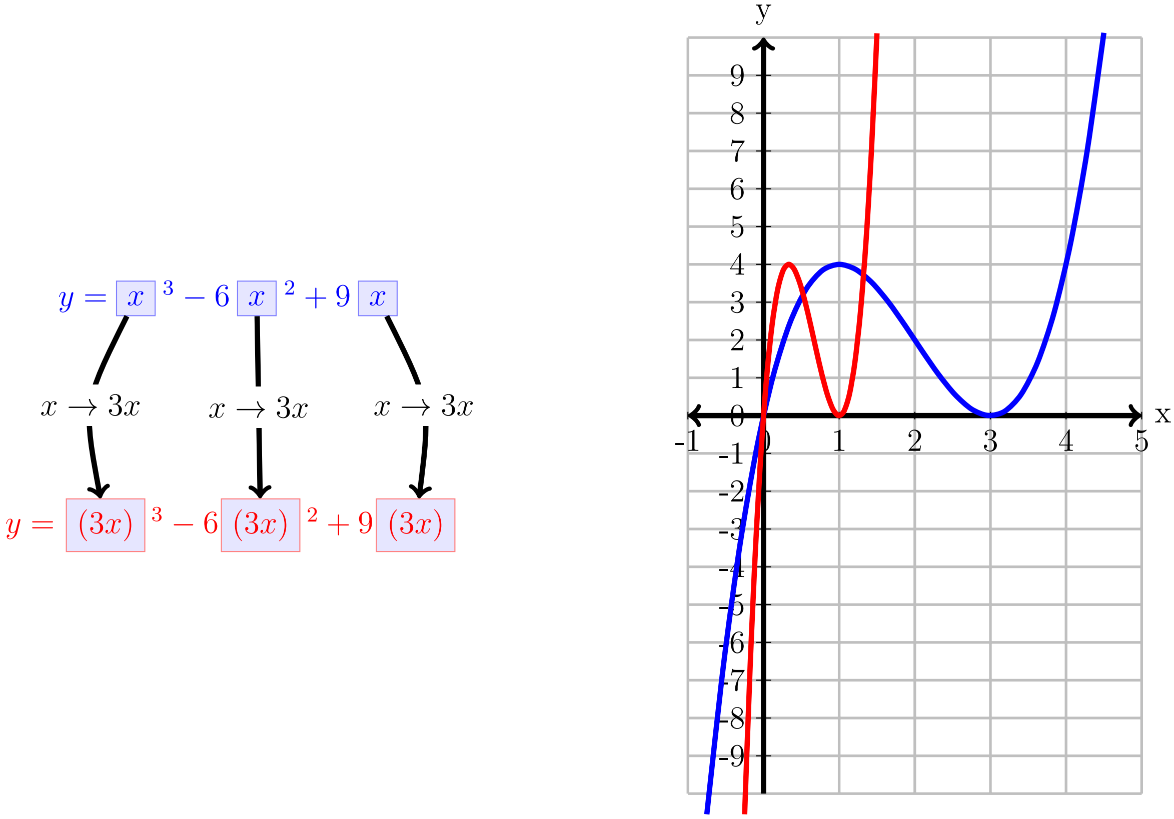
by compressing in x, scale factor 3.

Do you remember what equation transformation was equivalent to an x-compression, scale factor 3? It's:

$$x \to 3x \tag{4}$$

Again, we have more than one x in equation (3). So what do we do? We transform *every* x in equation (3) by the transformation (4): Have a look at Figure 15 for the solution:

Figure 15: Scaling $y = x^3 - 6x^2 + 9x$ by Compressing in x, Scale Factor 3



So the transformed equation will be:

$$y = (3x)^3 - 6(3x)^2 + 9(3x)$$

(which you could again simplify by multiplying out the brackets if you really want to. I'm not going to bother!).

4.3 Example 3 : Scaling $y = x^3 - 6x^2 + 9x$ by Stretching in the y-direction, Scale Factor 2

So here's our third example. This time we want to transform the graph of:

$$y = x^3 - 6x^2 + 9x \tag{5}$$

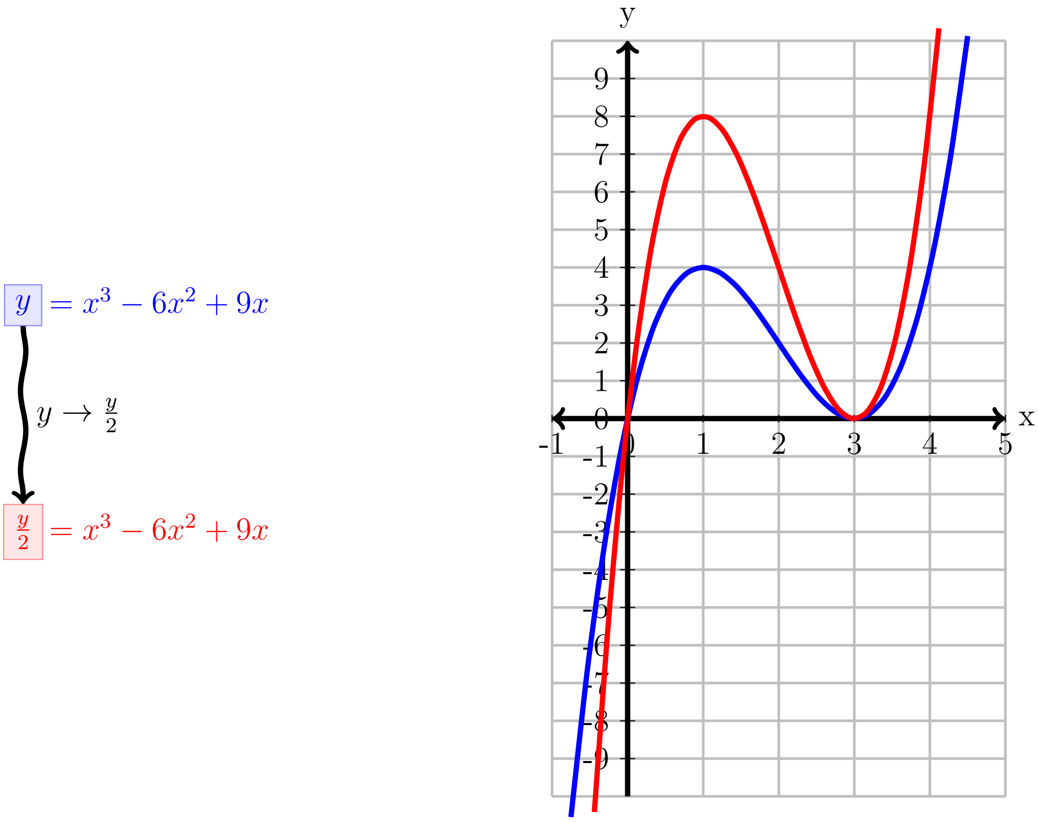
by stretching in y, scale factor 2.

Do you remember what equation transformation was equivalent to a y-stretch, scale factor 2? It's:

$$y \to \frac{y}{2} \tag{6}$$

This time there is only one y in equation (5). So we just transform it using the transformation (6): Have a look at Figure 16 for the solution:

Figure 16: Scaling $y = x^3 - 6x^2 + 9x$ by Stetching in y, Scale Factor 2



So the transformed equation will be:

$$\frac{y}{2} = x^3 - 6x^2 + 9x$$

which you could simplify this time to

$$y = 2(x^3 - 6x^2 + 9x)$$

5 Transforming an Actual Function 2: from Equation to Graph

Now in this section we're going to be looking at the reverse problem: when you know what the transformed equation is, how do you work out what the graph transformation is?

5.1 Example 4 : Finding the Transformation of $y = x(x + 1)$ to $y = 3x(3x + 1)$

In this example, we know what the equation transformation is: the original equation was

$$y = x(x + 1)$$

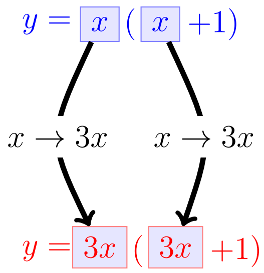
and it has been transformed to

$$y = 3x(3x + 1)$$

So how do you figure out what the graph transformation is?

Well the idea is to look for what has happened to the x and the y variables.

Figure 17: Finding the Transformation of $y = x(x + 1)$ to $y = 3x(3x + 1)$



Well, looking at Figure 17 we can see that nothing has happened to the y coordinate, but that wherever we had an x in the original equation, we have $3x$ in the transformed equation. So it's the x that has been transformed. So the graph transformation must be an x-thing; and since it is $x \to 3x$, that must be a compression, scale factor 3 (once I've had a look at Table 9 to remind myself!).

5.2 Example 5 : Finding the Transformation of $y = x(x+1)$ to $y = (x-3)(x-2)$

In this example, we know what the equation transformation is: the original equation was

$$y = x(x+1)$$

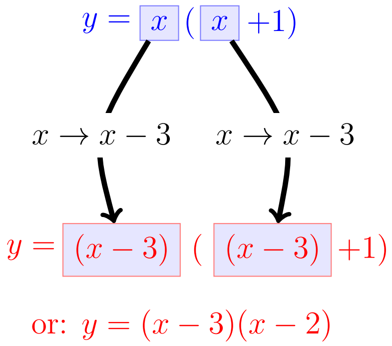
and it has been transformed to

$$y = (x-3)(x-2)$$

So how do you figure out what the graph transformation is?

Well the idea is to look for what has happened to the x and the y variables.

Figure 18: Finding the Transformation of $y = x(x+1)$ to $y = (x-3)(x-2)$



And looking at Figure 18 we can see that again, nothing has happened to the y coordinate, but that wherever we had an x in the original equation, we have $x-3$ in the transformed equation. So again it's the x that has been transformed. So the graph transformation must be an x-thing; and since it is $x \to x-3$, that must be a shift, to the right, by 3 (once I've had a look at Table 9 to remind myself!).

5.3 Example 6 : Finding the Transformation of $y = f(x)$ to $y = f(x - 3)$

Now in this example, we don't actually know what the original equation was. All we know is that it is given by $y = f(x)$. Does that mean we can't figure out what the graph transformation will be? Not a bit of it! Watch!

We know that the original function is:

$$y = f(x)$$

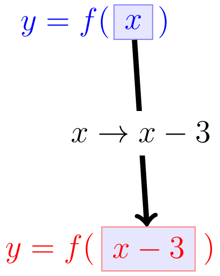
and it has been transformed to:

$$y = f(x - 3)$$

So how do you figure out what the graph transformation is?

Again the idea is to look for what has happened to the x and the y variables.

Figure 19: Finding the Transformation of $y = f(x)$ to $y = f(x - 3)$



Well, looking at Figure 19 we can see that nothing has happened to the y coordinate, but that wherever we had an x in the original equation, we have $x - 3$ in the transformed equation. So it's the x that has been transformed. So the graph transformation must be an x-thing; and since it is $x \to x - 3$, that must be a shift, to the right, by 3 (once I've had a look at Table 9 to remind myself!).

5.4 Example 7 : Finding the Transformation of $y = f(x)$ to $y = 3f(x)$

In this example, we know what the equation transformation is: the original equation was

$$y = f(x)$$

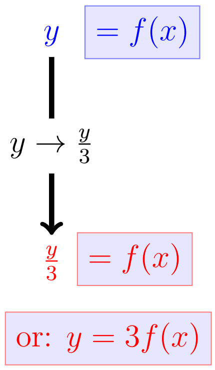
and it has been transformed to

$$y = 3f(x)$$

So how do you figure out what the graph transformation is?

Well the idea is to look for what has happened to the x and the y variables.

Figure 20: Finding the Transformation of $y = f(x)$ to $y = 3f(x)$



This time, it's not so obvious whether this is an x-thing or a y-thing. But, looking at Figure 20 we can see that by writing $y = 3f(x)$ as $\frac{y}{3} = f(x)$ then it's clearer that something has happened to the y coordinate (and nothing has happened to the x coordinate), and that wherever we had an y in the original equation, we have $\frac{y}{3}$ in the transformed equation. So it's the y that has been transformed. So the graph transformation must be a y-thing; and since it is $y \to \frac{y}{3}$, that must be a stretch, scale factor 3 (once I've had a look at Table 9 to remind myself!).

6 Doing Several Transformations

Right. So here's about a difficult as this stuff gets: what if we wanted to tranaform a graph *twice*? Say that we wanted to transform a graph by first shifting it 2 to the right, but then transforming *that* graph by compressing it in the x-direction, scale factor 3? Any ideas?

Well I'm hoping that you may have an inkling as to what to do! In fact, all you need to do is one transformation after the other. Watch this.

6.1 Example 8 : Shifting to the Right AND THEN Compressing in x

So here's our first example. Let's say that we wanted to transform the graph of:

$$y = x^3 - 6x^2 + 9x \tag{7}$$

by shifting it 2 units to the right, and then compressing that graph in x, scale factor 3. We just do one transformation after the other. First, the equation transformation that's equivalent to an x-shift to the right by 2 is:

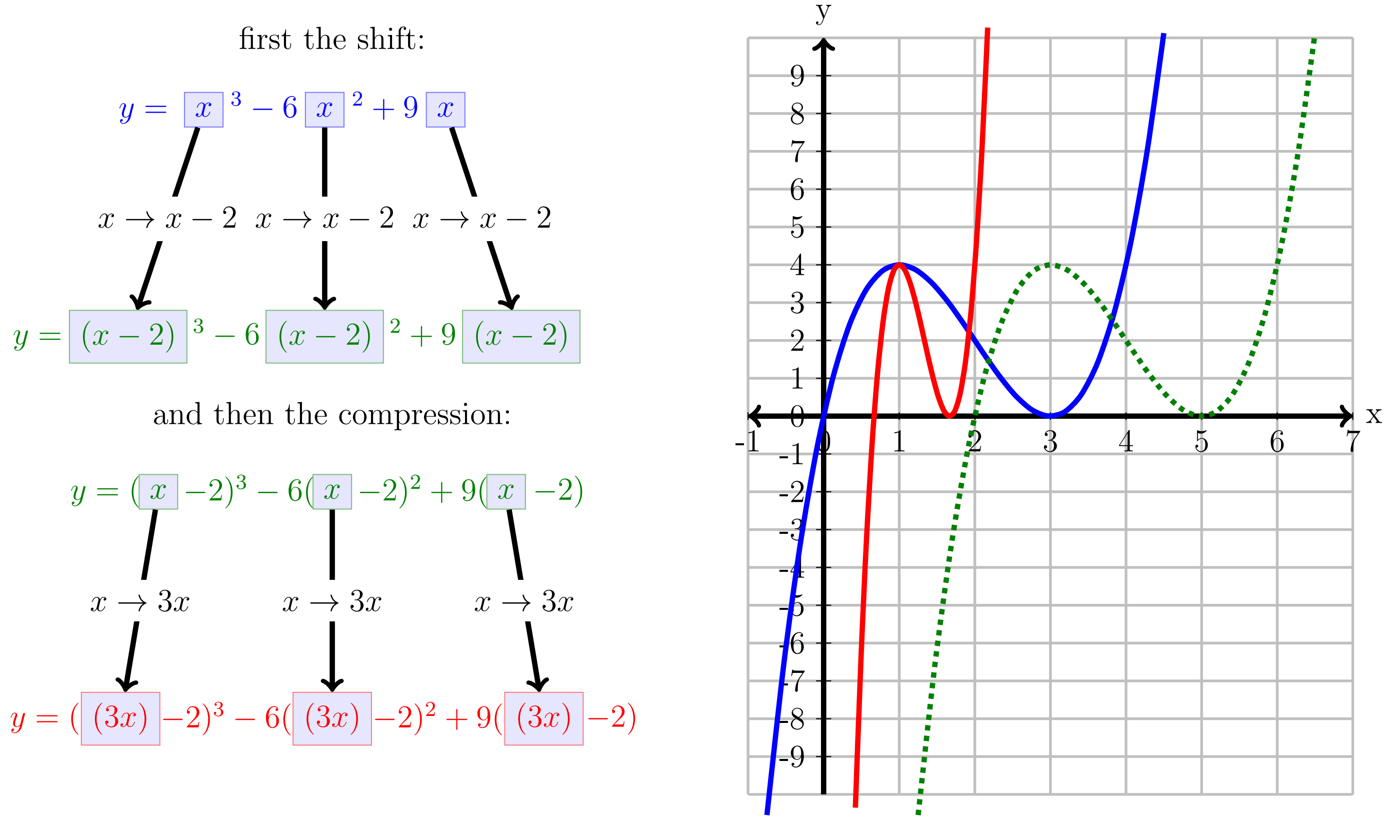
$$x \to x - 2 \tag{8}$$

And the equation transformation that's equivalent to an x-compression scale factor 3 is:

$$x \to 3x \tag{9}$$

Let's do it! Have a look at Figure 21 for the solution.

Figure 21: Shifting $y = x^3 - 6x^2 + 9x$ Two Units to the Right AND THEN Compressing in x, Scale Factor 3



So the equation will be transformed first from

$$y = x^3 - 6x^2 + 9x$$

to

$$y = (x-2)^3 - 6(x-2)^2 + 9(x-2)$$

by the shift, then this equation is transformed to

$$y = ((3x)-2)^3 - 6((3x)-2)^2 + 9((3x)-2)$$

by the compression. This last equation you could simplify by multiplying out the brackets if you really want to. I'm not going to bother, because I want to show the form of the final thing. I want you to see what's happened to the x during each transformation of the equation. In each case you figure out what the equation transformation should be, and apply it to the appropriate coordinate.

A Confused.com

When I was at school and I was learning about graph transformations, I used to get really confused. And the reason for that was that the teachers and the books used to say that if you wanted to transform $y = x^2$ to the *right* by 1 you'd get

$$y = (x - 1)^2$$

with a *negative* sign. But if you wanted to shift a graph *up* (the same kind of thing, but in the y-direction), then there would be a *positive* sign:

$$y = x^2 + 1$$

For me, that made no sense. Surely you ought to do the same thing for both x- and y-shifts?

Another thing they said was that if you were transforming the general function

$$y = f(x)$$

then if the transformation is an x-thing, the change goes in the bracket, and if it's a y-thing it goes outside the bracket. So an x-shift right by 1 would be

$$y = f(x - 1)$$

whereas a y-shift up by 1 would be

$$y = f(x) + 1$$

"remembering" of course that one is negative and the other is positive!!

Well you can forget all this twaddle! It was only until I realised that once you write

$$y = x^2 + 1$$

as

$$y - 1 = x^2$$

then you have made the situations consistent! You do the same thing to y as you do to x. Magic!

B The Size Of Scaling Factors

There's an extra complication that arises when you are doing scalings. And it's to do with the size of the scaling factor. I'll give you a for instance. Let's say that we were transforming the function $y = x^2$ by compressing it in x by a factor of 3. We now know (I hope!) that the equation transformation will yield

$$y = (3x)^2$$

OK. But what would happen if I chose the "compression" factor to be $\frac{1}{3}$ instead? Well, remember that for a "compression", we multiply by the scale factor (see Table 9), so we would get

$$y = \left(\frac{x}{3}\right)^2$$

But hang on a minute, isn't this transformation an *x-stretch*, scale factor 3? Yes it is. And the problem of course is the size of the scaling factor. In Tables 9 and 10, I have assumed that the scaling factor is greater than 1. A *compression* with a scale factor of less than 1 would be a *stretch*, and vice-versa.

So you've got to be a bit careful here! But I hope you get the idea!

References

Smith, S. (2013). Function Transformations II. Explains how equations and graphs are transformed with reflections and modulus operations.