Common Mistakes Students Make in A-Level Maths, and How to Prevent Them

Steve Smith, 1 February 2018

Contents

Prerequisites

None.

Notes

Quite unashamedly, these notes make extensive use of [Dawkins(2006)]. Also of great use (interest and mirth) were: [Maxwell(2006)], [Schechter(2009)] and [Bradis et al.(2016)Bradis, Minkovskii and Kharcheva].

Document History

Date	Version	Comments
1 February 2018	1.0	Initial creation of the document.

References

[Bradis et al.(2016)Bradis, Minkovskii and Kharcheva] **Bradis, V. M., Minkovskii, L. and Kharcheva, A. K.** (2016). *Lapses in Mathematical Reasoning*. Courier Dover Publications.

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[Schechter(2009)] **Schechter, E.** (2009). The Most Common Errors in Undergraduate Mathematics. website. `https://math.vanderbilt.edu/schectex/commerrs/`.

[Smith(2016a)] **Smith, S.** (2016a). Integration by Parts, the Tabular Method I: "DIS is how you do it!". Website. `http://www.chelseas-roost.co.uk/resources/maths/DIS_I.pdf`.

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Part I

The Kinds of Mistakes People Make...

1 Introduction

This is my first stab at trying to compile a list of common errors that people make when trying to answer A-Level Maths questions. I've a feeling that this document is just going to grow and grow and grow...

2 Algebra Errors

The topics covered here are mostly errors that students often make doing algebra, although some of the examples involve calculus (differentiation and integration) as well. And that's because you hardly ever do algebra in isolation: it's usually a part of doing something else.

I'm convinced that many of the mistakes given here are caused by people getting lazy or being in a hurry and not paying attention to what they're doing. By slowing down, paying attention, and using proper notation, you can avoid the vast majority of these mistakes!

2.1 Division by Zero

Everyone knows that $\frac{0}{2} = 0$, but the problem is that far too many people also think that $\frac{2}{0} = 0$ or $\frac{2}{0} = 2$!

Well, they're not! *Division by zero is undefined*! You simply cannot divide by zero, so you mustn't do it!

Here is a good example of the kind of havoc that can arise when you divide both sides of an equation by zero. See if you can find the mistake that I make in this nifty bit of thinking:

Example 1: Proving $1 = 2$

$a = b$	1 : Let's start by assuming that this is true
$ab = a^2$	2 : Multiply both sides by a
$ab - b^2 = a^2 - b^2$	3 : Subtract b^2 from both sides
$b(a-b) = (a+b)(a-b)$	4 : Factorise both sides (using *the difference of two squares* on the RHS)
$b = a + b$	5 : Divide both sides by $a - b$
$b = 2b$	6 : Remember that we started off by assuming that $a = b$
$1 = 2$	7 : Divide both sides by b

So, we've managed to prove that $1 = 2$! Now, we know that that's not true (hopefully) so clearly we have made a mistake somewhere. Can you see where?

The mistake occurred going from line 4 to line 5. Remember that we started out with the assumption that $a = b$. However, if this is true then $a - b = 0$! So, in going from line 4 to line 5 we are dividing both sides by zero!

That simple mistake led us to something that we knew wasn't true. In a lot of cases when you divide both sides of an equation by zero, your answer will not be obviously wrong, so it will be much harder to spot your mistake. It will not always be clear that you are dividing by zero, as was the case in this example. Always be on the lookout for this kind of thing.

Whenever you divide both sides of an equation *by a variable*, you should ask yourself "Could this variable ever be zero?" If not, fine. But if it could, bad things are almost guaranteed to happen...

We're going to encounter this problem throughout this document. See particularly Section 2.5.

Remember that *you CAN'T divide by zero*!

2.2 Bad/Lost/Assumed Brackets

This kind of mistake is one of the most frustrating. In this category, there are a couple of errors that people are commonly prone to make.

2.2.1 Missing Brackets

The first error is that people get lazy and decide that brackets aren't needed at certain steps, or that they can remember that the brackets are supposed to be there, but don't actually write them in. Of course, the problem is then that they often forget about the-missing-brackets-that-should-be-there in the very next step!

Here's an example of this missing-brackets thing:

Example 2: Subtract $4x - 5$ from $x^2 + 3x - 5$

Here's the right way to do it:

$$\begin{aligned}(x^2 + 3x - 5) - (4x - 5) &= x^2 + 3x - 5 - 4x + 5 \\ &= x^2 - x\end{aligned}$$

and here's the wrong way:

$$x^2 + 3x - 5 - 4x - 5 = x^2 - x - 10$$

Be careful and note the difference between these two! In the first case I've put brackets around the $x^2 + 3x - 5$ and the $4x - 5$, and in the second case I didn't. Since we are subtracting more than one term (the $4x$ and the -5) we need to make sure we subtract the *whole expression* of $4x - 5$! The only way to make sure we do that correctly is to put brackets around it.

So this is one of those errors where people know that technically the brackets should be there (I hope), but they don't put them in and promptly forget that they were there and do the subtraction incorrectly.

Here's another example:

Example 3: Evaluate $-3\int 6x - 2\,dx$

Here's the right way to do it:

$$\begin{aligned}-3\int 6x - 2\,dx &= -3\left(3x^2 - 2x + C\right) \\ &= -9x^2 + 6x + D\end{aligned}$$

and here's the wrong way:

$$\begin{aligned}-3\int 6x - 2\,dx &= -3 \times 3x^2 - 2x + C \\ &= -9x^2 - 2x + C\end{aligned}$$

Note the use of the brackets. The problem states that what is required is -3 times the *whole* integral, not -3 times only the first term of the integral (as is done in the incorrect example).

2.2.2 Brackets are Important

The other error is that students sometimes don't understand just how important brackets really are. This is often seen in errors made in taking powers of numbers, as the next few examples show.

Example 4: Raising a Quantity to Some Power

Let's say we had the quantity $4x$ and we needed to square it. That is, raise it to a power of 2. Here's the right way to do it:

$$(4x)^2 = (4)^2(x)^2 = 16x^2$$

and here's the wrong way:

$$4x^2$$

Note the very important difference between these two! When dealing with powers remember that *only the quantity immediately to the left of the power is the thing that the power applies to.*

So, in the incorrect case above, the x is the quantity immediately to the left of the power, so we are squaring only the x while the 4 isn't squared. In the correct case the brackets are immediately to the left of the power so this signifies that everything inside the brackets should be squared!

So

$$(4x)^2 \neq 4x^2$$

Brackets are required in this case to make sure we square the whole thing, not just the x, so don't forget them!

Example 5: Square -3

Let's say we had the quantity -3 and we needed to square it. That is, raise it to a power of 2. Here's the right way to do it:

$$(-3)^2 = (-3) \times (-3) = 9$$

and here's the wrong way:

$$-3^2 = -(3)(3) = -9$$

This one is similar to the previous one, but has a subtlety that causes problems on occasion. Remember that *only the quantity to the left of the power is the thing that the power applies to*. So, in the incorrect case *only the* 3 is to the left of the power and so *only the* 3 gets squared!

Many people know that technically they are supposed to square the whole of the -3, but they get lazy and don't write the brackets in on the premise that they will remember them when the time comes to actually evaluate the thing. However, it's amazing how many of these same people promptly forget about the-missing-brackets-that-should-be-there and write down -9 anyway!

Here's another example:

Example 6: Convert $\sqrt{7x}$ to Fractional Powers

Here's the right way to do it:

$$\sqrt{7x} = (7x)^{\frac{1}{2}} = 7^{\frac{1}{2}}x^{\frac{1}{2}}$$

and here's the wrong way:

$$\sqrt{7x} = 7x^{\frac{1}{2}}$$

This comes back to the same mistake as in a previous example. That only the quantity to the left of the power has the power applied to it. So, the incorrect case is really $7x^{\frac{1}{2}} = 7\sqrt{x}$ which is clearly *not* the original root. In the original root, the whole of the $7x$ is being square-rooted.

2.3 Poor Distribution (Multiplying Brackets Out)

Be careful when multiplying brackets out! There two main errors that I come across on a regular basis.

2.3.1 Multiply *Everything* Inside the Brackets by the Coefficient

The most common of these errors is the failure to multiply everything inside the brackets by the coefficient on the outside. For example:

Example 7: Multiply out $4(2x^2 - 10)$

Here's the right way to do it:

$$\begin{aligned} 4(2x^2 - 10) &= 4 \times 2x^2 - 4 \times 10 \\ &= 8x^2 - 40 \end{aligned}$$

and here's the wrong way:

$$4(2x^2 - 10) = 8x^2 - 10$$

Make sure that you distribute the coefficient all the way through the brackets! Too often people just multiply the first term by the coefficient and ignore the rest.

This is especially true when the second term is just a number. For some reason, if the second term contains variables students will remember to do the distribution correctly more often than not. Strange.

2.3.2 Powers Take Precedence Over Distribution

What that means in English is that if your brackets are raised to some power, then you have to multiply out the brackets first, before you multiply the result by the coefficient. For example:

Example 8: Multiply out $3(2x - 5)^2$

Here's the right way to do it (squaring the brackets first):

$$\begin{aligned} 3(2x-5)^2 &= 3(4x^2 - 10x - 10x + 25) \\ &= 3(4x^2 - 20x + 25) \\ &= 12x^2 - 60x + 75 \end{aligned}$$

and here's the wrong way (distributing the 3 first):

$$\begin{aligned} 3(2x-5)^2 &= (6x - 15)^2 \\ &= 36x^2 - 90x - 90x + 225 \\ &= 36x^2 - 180x + 225 \end{aligned}$$

Remember that from *BODMAS*, evaluating the power must be performed *before* you distribute any coefficients through the brackets!

Brackets mean *do me first*!

2.4 Bad Assumptions Concerning Operations

First up: what do I mean by an "operation"? Well, an operation is a thing that you do to a number, or a function. So *addition* is an operation; so is *multiplication*. And *division*, *taking the reciprocal*, *squaring*, *square-rooting*, *taking the sine of*, *differentiating*, and *integrating*. There are lots of others!

Next up: an apology! I didn't know what else to call this section, so I apologise for the naff title. But there's an error here that often crops up, so I wanted to make sure I covered it.

Here's the bad assumption: since $2(x+y) = 2x+2y$, then that idea can be used in lots of other situations. However, there is a whole list of stuff that *doesn't* work like this. For instances:

Examples 9: Bad Additive Assumptions

$$(x+y)^2 \neq x^2 + y^2$$
$$\sqrt{x+y} \neq \sqrt{x} + \sqrt{y}$$
$$\frac{1}{x+y} \neq \frac{1}{x} + \frac{1}{y}$$
$$\cos(x+y) \neq \cos(x) + \cos(y)$$

Its not hard to convince yourself that these aren't true. Just pick a couple of numbers and plug them in! For instance,

$$(1+3)^2 \stackrel{?}{=} 1^2 + 3^2$$
$$\implies (4)^2 \stackrel{?}{=} 1+9$$
$$\implies 16 \stackrel{?}{=} 10$$

You will find the occasional set of numbers for which one of these rules will work, but they don't work *in general*.

Note that there are far more examples where this additive assumption doesn't work than I've listed here. I simply wrote down the ones that I see most often. Also, a couple of those that I listed could be made more general. For instance,

$$(x+y)^n \neq x^n + y^n \qquad \text{for any integer } n \geq 2$$
$$\sqrt[n]{x+y} \neq \sqrt[n]{x} + \sqrt[n]{y} \qquad \text{for any integer } n \geq 2$$

So the big picture here is that operations behave in different ways. What works for some operations, doesn't work for others.

2.5 Cancelling Errors

These errors fall into two categories: simplifying rational expressions; and solving equations.

That's because cancelling is generally found in two situations: when you have a fraction, and you want to cancel something from top and bottom; and when you have an equation, and you want to cancel something from both sides (that is, divide both sides of your equation by something).

2.5.1 Simplfying Rational Expressions

Let's look at simplifying rational expressions first. What is a *rational expression*? A rational expression is just a fraction with algebra in it. The word *ratio*nal is derived from the word *ratio*, and ratios can be expressed as fractions.

Here's an example:

Example 10: Simplify $\dfrac{3x^3 - x}{x}$

Here's the right way to do it:

$$\begin{aligned}
\frac{3x^3 - x}{x} &= \frac{x(3x^2 - 1)}{x} && \text{factoring out an } x \text{ on the top} \\
&= \frac{x}{x} \times \frac{3x^2 - 1}{1} && \text{using the way that fractions multiply} \\
&= 1 \times \frac{3x^2 - 1}{1} && \text{anything divided by itself is 1} \\
&= \frac{3x^2 - 1}{1} && \text{multiplying by 1 leaves a quantity unchanged} \\
&= 3x^2 - 1 && \text{dividing by 1 leaves a quantity unchanged}
\end{aligned}$$

Now I don't advocate doing all these steps when you cancel something from the top and the bottom of a fraction. The reason I've gone to all this trouble is that in my experience, students *don't know why cancelling works*. This means they don't really know when to apply it, and when *not* to apply it, and quite often get it wrong.

To show a couple of ways in which it is possible to get cancelling wrong, have a look at:

$$\frac{3x^3 - x}{x} = 3x^2 - x$$

where only the first term on the top is being cancelled with the x on the bottom, and

$$\frac{3x^3 - x}{x} = 3x^3 - 1$$

where the x on the bottom is only cancelled with the second term on the top.

As seen above, *cancelling depends on the way that fractions multiply*. This is a very important idea!

So actually, the only time you can cancel something from top and bottom of a fraction is when:

- that something is *multiplying* everything else on the top (so you could factorise it out of the top),
- *and* that *same* something is *multiplying* everything else on the bottom (so you could factorise it out of the bottom).

Both of these conditions have to be met. If not, you can't cancel the something.

2.5.2 Solving Equations

Now, let's take a quick look at cancelling errors involved in solving equations. That is, problems that arise from dividing both sides of an equation by something.

Here's an example:

Example 11: Solve $2x^2 = x$

First, here's how *not* to do it. The biggest mistake in solving this kind of equation is to cancel an x from both sides (by dividing both sides by x), so that

$$\begin{aligned} 2x^2 &= x \\ \implies 2x &= 1 \\ \implies x &= \tfrac{1}{2} \end{aligned}$$

While, $x = \frac{1}{2}$ *is* a solution to the above equation, *there is another solution that we've missed*. Can you spot it?

Check this out. Starting again, get everything on one side of the equation by subtracting x from both sides:

$$2x^2 - x = 0$$

then factorize:

$$x(2x - 1) = 0$$

Because here we have two things (the x and the $2x - 1$) that multiply together to give zero, then *one of them must be zero*. From this we can see that either $x = 0$ *or* $2x - 1 = 0$.

From $2x - 1 = 0$ we get the $x = \frac{1}{2}$ that we got in the first attempt above, but from $x = 0$ we also get the other solution that we *didn't* get in the first attempt.

Clearly $x = 0$ will make the original equation work and so is a solution!

We missed the $x = 0$ in the first attempt because we tried to make our life easier by "simplifying" the equation before solving it. And in the process we divided both sides of an equation by something that could be zero.

While some simplification is a good and necessary thing, you should *never* divide both sides of an equation by a variable that could be zero. If you do this, you *will* lose solutions (at best. And at worst you could end up with complete crap!).

This is so important that I'm going to go over it again. The reason you can lose solutions when solving equations is that if you do this:

$$\begin{aligned} 2x^2 &= x \\ \implies 2x &= 1 \end{aligned}$$

what are you doing to both sides? You are dividing both sides by x. But we don't know what the value of x is. So it could be zero! Which means that you could be dividing both sides of this equation by zero!

If you were completely sure that your unknown thing could not possibly be zero, then you are quite justified in dividing both sides of an equation by it. Here's an example of when you *can* do this.

Example 12: Solve $2\cos^2(x) = \cos(x)$, for $0 < x < \frac{\pi}{2}$

Now in the range $0 < x < \frac{\pi}{2}$, $\cos(x)$ is *always positive*, and so it can't be zero. That means we can quite happily divide both sides of the equation by it, and we won't lose any solutions:

$$\begin{aligned} 2\cos^2(x) &= \cos(x) \\ \implies 2\cos(x) &= 1 \\ \implies \cos(x) &= \tfrac{1}{2}, \text{ etc.} \end{aligned}$$

2.6 Proper Use of Square Root

There is a common misconception about the use of square roots. Students seem to believe that

$$\sqrt{16} = \pm 4$$

But this is not actually correct! Square roots are *always* positive (or zero)! So the correct value of $\sqrt{16}$ is

$$\sqrt{16} = 4$$

This is the *only* value of the square root! If we want a negative value then we would have to do something like this:

$$-\sqrt{16} = -\left(\sqrt{16}\right) = -(4) = -4$$

Notice that I used the brackets only to make the point that $\sqrt{16} = 4$.

I think that this misconception arises because we are often required to solve things like $x^2 = 16$. Clearly the answer to this is $x = \pm 4$ and often students will solve this by "taking the square root" of both sides. There is a missing step however, that everyone leaves out (me included), that gives rise to the misconception.

Here is the proper solution technique for this problem:

Example 13: Solving $x^2 = 16$

$$\begin{aligned} x^2 &= 16 \\ \implies x &= \pm\sqrt{16} \\ \implies x &= \pm 4 \end{aligned}$$

Note that the $\pm$ shows up in the second step *before* we actually find the value of the square root! It doesn't show up as part of the taking of the square root.

Many instructors (including myself) don't help matters in that they will often omit the second step and by doing so seem to imply that the $\pm$ is showing up because of the taking-the-square-root thing.

So, remember that square roots *always* return a positive answer (or zero).

2.7 Ambiguous Fractions

This is more a notational issue than an algebra issue. There are really three kinds of "bad" notation that students often use with fractions that can lead to errors in their work.

2.7.1 Using the Correct Division Symbol

The first is using a "/" to denote a fraction. $2/3$, for instance. In this case there really isn't a problem with using a "/", but what about $2/3x$? This can be either of the two following fractions:

$$\frac{2}{3}x \qquad \text{or} \qquad \frac{2}{3x}$$

It is not clear from $2/3x$ which of these two it should be! You, as the student, may know which one of the two that you intended it to be, but an examiner won't. And while you may know which of the two you intended it to be when you wrote it down, will you remember on the very next line of working? And will you still know which of the two it is when you go back to look at the problem when you use these notes to revise for an exam?

You should only use a "/" for fractions when it will be clear and obvious to everyone, not just you, how the fraction should be interpreted. But far best would be to *never* use "/", and only use horizontal lines to denote fractions. Then there will never be any doubt.

2.7.2 Careless Placement of Items

The next notational problem I see fairly regularly is people writing things like

$$\frac{2}{3}\,x$$

It's not clear from this if the x belongs in the denominator of the fraction or not, because the x isn't actually under the horizontal division line. Students often write fractions like this and usually they mean that the x shouldn't be in the denominator. The problem is on a quick glance it often looks like it should be in the denominator and the student just didn't draw the fraction bar over far enough.

If you intend for the x to be in the denominator then write it as such that way, $\frac{2}{3x}$, i.e. make sure that you draw the fraction bar over the *whole* denominator. If you don't intend for it to be in the denominator then don't leave any doubt! Write it as $\frac{2}{3}x$.

2.7.3 Using the Correct Division Symbol Revisited

The final notational problem that I see comes back to using a "/" to denote a fraction, but is really a brackets problem. This involves fractions like

$$\frac{a+b}{c+d}$$

Often students who use "/" to denote fractions will write this fraction as

$$a+b/c+d$$

These students know that they are writing down the original fraction. However, almost anyone else will see the following

$$a+\frac{b}{c}+d$$

This is definitely *not* the original fraction. So, if you *must* use "/" to denote fractions, use brackets to make it clear what is the numerator and what is the denominator. So, you should write it as

$$(a+b)/(c+d)$$

But why get yourself into all this trouble? Just write all fractions with horizontal bars instead of diagonal ones!!

3 Function Errors

3.1 $\cos x$ is *not* a Multiplication...

I see students on a regular basis believing that such things as these are true:

$$\cos(x+y) \neq \cos(x)+\cos(y)$$
$$\cos(3x) \neq 3\cos(x)$$

These just simply aren't true. If you're not sure you believe that those aren't true just pick a couple of values for x and y and plug them in. Try $x = \pi$ and $y = 2\pi$ in the first equation, for instance.

There are two possible reasons for thinking this kind of thing. One is the "Bad Assumptions Concerning Operations" problem (see Section 2.4). That's bad enough.

But the other reason is much more worrying. Students sometimes (but only very rarely, thankfully) think of $\cos x$ as a *multiplication* of something called cos and something called x. I did have a student once who tried to solve the equation

$$\cos x = \frac{1}{2}$$

by dividing both sides by cos!!!

$\cos x$ as being a product of two things just couldn't be farther from the truth! Cosine is a *function* and cos is the *name* of the function! (Or at least, a shorthand for the full name of the function, which is "cosine x").

There is a glimmer of an excuse that students could use as a defence for making this kind of mistake. And that is the wanton disregard by almost all teachers, books, published papers, and examination papers (in fact the whole mathematics industry) to use proper function notation when the functions are trigonometrical ones, or logarithms. So, for example, in almost every book you come across you will see

$$\sin x \qquad \text{or} \qquad \cos x \qquad \text{or} \qquad \ln x$$

when they should really be written as

$$\sin(x) \qquad \text{or} \qquad \cos(x) \qquad \text{or} \qquad \ln(x)$$

where the function notation is emphasised. I mean, when introducing a function f, a book wouldn't write

$$fx = x^2 + 1$$

would it?

3.2 ...So Use Proper Function Notation to Emphasise It

Proper function notation has the name of the function, followed by a pair of brackets that surround the *argument* of the function.

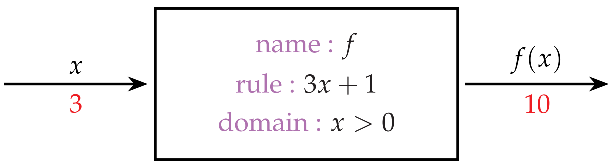

Figure 1: An Example of a Function (With an Example "Input" and "Output")

Figure 1 shows the idea of a function. A function is like a black box with an "in" slot and an "out" slot. A number gets pushed into the "in" slot, and immediately another number comes out of the "out" slot. What the function does is to transform the in-number according to some *rule* (in this case $x \to 3x+1$), and the transformed number is what comes out of the "out" slot. This is a much bigger idea than that of simple multiplication.

Functions are *transformers* of numbers. They perform *operations* on them. So always use the proper function notation, *including the brackets*, whenever you are dealing with a function, to show clearly that you are dealing with a function.

4 Trigonometry Errors

4.1 Degrees and Radians

When you start learning trigonometry, everything is done using degrees as the unit for measuring angles. Even at A-Level, trigonometry is often taught using degrees. And exam questions often involve degrees, rather than radians. But the problem is that *almost all calculus is done in radians*. In fact, the biggest reason for having radians to measure angles is so that calculus involving trigonometrical functions is easier.

You simply *must* get used to doing everything in radians in trigonometry, so that you are just as comfortable using radians as you are using degrees.

Because there are two different units for measuring angles, you have to be really careful which unit you are using in a given question. So, if you are asked to evaluate $\cos(x)$ at $x = 10$, are you being asked to use 10 radians or 10 degrees? The answers are very, very different! Consider the following:

$$\cos(10) \approx -0.839071529076 \text{ (using 10 radians)}$$
$$\cos(10) \approx 0.984807753012 \text{ (using 10 degrees)}$$

They don't even have the same sign! So, make sure that you always know whether you are using degrees or radians when dealing with a given trigonometry question. And if you are doing calculus with your trigonometrical functions, *angles will always be in radians*. And make sure your calculator is set to do calculations in the right units!!

4.2 Powers of Trigonometrical Functions

Remember that if n is a positive integer then

$$\sin^n(x) \text{ means } \Big[\sin(x)\Big]^n$$

The same holds for all the other trigonometrical functions as well of course. This is just a notational idiosyncrasy that you've got to get used to. Also remember to keep the following straight:

$$\tan^2 x \qquad \text{and} \qquad \tan x^2$$

In the first case we are taking the tangent then squaring the result, and in the second we are squaring the x then taking the tangent.

This would never be a problem of course if teachers and books, etc, used the proper function notation when using trigonometrical functions (and others too). See Section 3.2. If we wrote

$$\tan^2(x) \qquad \text{and} \qquad \tan(x^2)$$

then that would remove all the ambiguity. But we don't. So you'll have to be prepared to tackle this problem when you come across it. But *you* should always use the proper function notation, with the brackets around the argument.

4.3 Inverse Trigonometrical Function Notation

The notation for inverse trigonometrical functions is also not the best. You need to remember that despite what I've just said above,

$$\cos^{-1}(x) \neq \frac{1}{\cos(x)}$$

This is why I said that n had to be a *positive* integer in the previous discussion. I wanted to avoid this notational problem. The -1 in $\cos^{-1}(x)$ is *not* an exponent. It is there to denote the fact that we are dealing with the *inverse* trigonometrical function.

There is another notation for inverse trigonometrical functions that avoids this problem:

$$\text{“the inverse cosine function”}(x) \equiv \cos^{-1}(x) \equiv arc\cos(x)$$

but *arc* cos is not always used. And I'm not sure that it should be. *arc* cos is historically the name of the inverse cosine function, but now we have the modern function notation, we should really use that. The problem of course is that the modern function notation can be ambiguous. Mathematicians, eh?[1]

[1] See Appendix A for more interesting and ambiguous mathematical notation!

5 Calculus Errors

Many of the errors listed here are not really calculus errors, but errors that commonly occur during calculus questions, and are really notational errors that are calculus related.

5.1 Derivatives and Integrals of Products and Quotients

5.1.1 Bad Assumptions Concerning Operations, Revisited

Remember bad assumptions concerning operations? Well, there are a few calculus-related ones:

Example 14: Bad Assumptions Concerning Operations in Calculus

Recall that while

$$(f \pm g)'(x) \equiv f'(x) \pm g'(x) \qquad \text{and} \qquad \int \Big[f(x) \pm g(x)\Big]\, dx \equiv \int f(x)\, dx \pm \int g(x)\, dx$$

are true, the same thing can't be done for products and quotients. In other words,

$$(fg)'(x) \neq f'(x)g'(x) \qquad \text{and} \qquad \int \Big[f(x)g(x)\Big]\, dx \neq \int f(x)\, dx \times \int g(x)\, dx$$

$$\text{and} \qquad \left(\frac{f}{g}\right)'(x) \neq \frac{f'(x)}{g'(x)} \qquad \text{and} \qquad \int \frac{f(x)}{g(x)}\, dx \neq \frac{\int f(x)\, dx}{\int g(x)\, dx}$$

If you need convincing of this consider the example of $f(x) = x^4$ and $g(x) = x^{10}$. Then

$$\begin{aligned} (fg)'(x) &\stackrel{?}{=} f'(x)g'(x) \\ (x^4x^{10})'(x) &\stackrel{?}{=} (x^4)'(x^{10})' \\ (x^{14})'(x) &\stackrel{?}{=} (4x^3)(10x^9) \\ 14x^{13}(x) &\stackrel{?}{=} 40x^{12} \end{aligned}$$

I only worked through the case of the derivative of a product, and in that case the two are clearly not equal. I'll leave it to you to check the remaining three cases if you'd like to (you should!).

Remember that in the case of derivatives we've got the product and quotient rules. In the case of integrals we have only the product rule (which, absurdly, is called *Integration by Parts*).

When faced with an integral of a quotient there is no general rule, and it will have to be dealt with on a case by case basis.

5.2 Proper Use of the Formula for $\int x^n \, dx$

Many students forget that there are two restrictions on this integration formula, so for the record here is the formula, along with the two restrictions:

Example 15: The Restriction on the Polynomial Integration Rule

$$\int x^n\, dx \equiv \frac{x^{n+1}}{n+1} + C, \qquad \text{provided } n \text{ is a constant, and } n \neq -1$$

I'll be mentioning the first of these two restrictions later (see Section 6.3) in regard to differentiation.

The second restriction (that $n \neq -1$) is incredibly important because if we allowed $n = -1$ we would get division by zero in the formula! Here is what I see far too many students do when faced with this integral:

$$\int x^{-1}\,dx \equiv \frac{x^0}{0} + C \equiv x^0 + C \equiv 1 + C$$

Well, where do you start with this? First there's the improper use of the formula, then there is the division by zero problem, and to finish off nicely, what's the point in writing $1 + C$ (if C can be *any* constant)?

Actually, the correct integral of x^{-1} is

$$\int x^{-1}\,dx \equiv \int \frac{1}{x}\,dx \equiv \ln(|x|) + C \tag{1}$$

which leads us nicely on to the next error.

5.3 Dropping the absolute value when integrating $\int \frac{1}{x}\,dx$

In formula (1), the absolute value bars on the argument are absolutely necessary (sorry about the pun)! Part of the reason for this is that all the $\log(x)$ functions are only defined for $x > 0$, so it's really important to have a positive argument to a log function.

But why can you just make it positive by putting modulus signs in? Putting modulus signs in will certainly make the argument positive, but are there legitimate reasons why you can do that? For an explanation, see Appendix B.

It is certainly true that on occasion the modulus signs can be dropped after the integration, but they are required in most cases. For instance:

Example 16: The Modulus Signs in $\int \frac{1}{x}\,dx = \ln(|x|) + C$

Contrast the two integrals,

$$\int \frac{2x}{x^2+10}\,dx \equiv \ln(|x^2+10|) + C \equiv \ln(x^2+10) + C, \text{ and}$$
$$\int \frac{2x}{x^2-10}\,dx \equiv \ln(|x^2-10|) + C$$

In the first case the x^2 is positive and adding 10 will keep it positive, so since $x^2 + 10 > 0$ (whatever the value of x) we can drop the absolute value bars. And this is because the argument to the log function is guaranteed to be positive.

In the second case however, since we don't know what the value of x is, there is no way to know the sign of $x^2 - 10$ and so the absolute value bars are required to ensure that the argument to the log function is positive.

As I've said, all this is necessary because the $\log(x)$ functions are only defined for $x > 0$. So we have to ensure that the arguments of these functions are all greater than zero!

5.4 Improper Use of the Formula $\int \frac{1}{x}\,dx$

This error is that of assuming that if $\frac{1}{x}$ integrates to $\ln(|x|)$ then $\frac{1}{\text{something}}$ integrates to $\ln(|\text{something}|)$! The following table gives some examples of where this doesn't work:

Integral	**Incorrect Answer**	**Correct Answer**
$\int \frac{1}{x^2+1}\,dx$	$\ln(x^2+1)+C$	$\tan^{-1}(x)+C$
$\int \frac{1}{x^2}\,dx$	$\ln(x^2)+C$	$-x^{-1}+C \equiv \frac{1}{x}+C$
$\int \frac{1}{\cos(x)}\,dx$	$\ln(\lvert\cos(x)\rvert)+C$	$\ln(\lvert\sec(x)+\tan(x)\rvert)+C$

So, be careful when attempting to use formula (1). This formula can only be used when the integral is of the form $\int \frac{1}{x}\,dx$. If your integral is not of this form, then you can't use (1). An integral can often be massaged into this form with an appropriate change of variable (the old u-substitution ploy). But if the integral can't be transformed into $\int \frac{1}{x}\,dx$ then the integral *cannot* be evaluated using (1).

However, this will be true:

$$\int \frac{1}{\text{something}}\,d\,\text{something} = \ln(|\text{something}|)+C$$

because it has *exactly* the form of Equation (1).

5.5 Improper Use of Integration Formulas in General

This one is really the same issue as the previous one, but so many students have trouble with logarithms that I wanted to treat that example separately.

So, as with the previous issue, students tend to try and use "simple" formulas that they know to be true on integrals that, on the surface, kind of look the same. So, for instance:

Example 17: Faulty Use of "Simple" Formulas

The following two formulas,

$$\int \sqrt{u}\,du \equiv \tfrac{2}{3}u^{\frac{3}{2}}+C, \text{ and}$$
$$\int u^2\,du \equiv \tfrac{1}{3}u^3+C$$

are correct. The mistake is to assume that if these are true then the following must also be true:

$$\int \sqrt{\text{anything}}\,du \equiv \tfrac{2}{3}\text{anything}^{\frac{3}{2}}+C, \text{ and}$$
$$\int \text{anything}\,du \equiv \tfrac{1}{3}\text{anything}^3+C$$

This just isn't true! The first set of formulas work because we have the square root of a single variable or a single variable squared. If there is anything other than a single u under the square root or being squared then those formulas won't work. On occasion these will hold for things other than a single u, but in general they won't, so be careful!

Here's another couple of examples of these formulas not being used correctly:

Integral	Incorrect Answer	Correct Answer
$\int \sqrt{x^2+1}\,dx$	$\frac{2}{3}(x^2+1)^{\frac{3}{2}}+C$	$\frac{1}{2}\left(x\sqrt{x^2+1}+\ln(\lvert x+\sqrt{x^2+1}\rvert)\right)+C$
$\int \cos^2(x)\,dx$	$\frac{1}{3}\cos^3(x)+C$	$\frac{1}{2}x+\frac{1}{4}\sin(2x)+C$

If you aren't convinced that the wrong answers really aren't correct then remember that you can always check your answers to indefinite integrals by differentiating the answer. If you differentiate correctly you should get the function you originally integrated, although in each case it will take some simplification to get the answers to be the same.

5.6 Bad Derivative Notation

Students can be very careless with derivative notation.

Example 18: Bad Derivative Notation

When asked to differentiate $f(x) = x(x^3-2)$, for example, I will often get this kind of answer:

$$f(x) = x(x^3-2) = x^4-2x = 4x^3-2$$

This is again a situation where you may know what you're intending to say, but anyone else who reads this will come away with the idea that $x^4-2x = 4x^3-2$ and that is clearly *not* what you are trying to say. However, it *is what you are saying when you write it this way*.

The proper notation is

$$\begin{aligned} & f(x) = x(x^3-2) = x^4-2x \\ \implies & f'(x) = 4x^3-2 \end{aligned}$$

5.7 Failure of Integration Notation

There are many notation errors that occur with integrals...

5.7.1 Loss of the Integration Sign

...quite a lot of them involving the integration sign itself: either putting it in the wrong place, or more commonly, not putting it where it needs to be.

Let's start with this example:

Example 19: Loss of the Integration Sign

$$\int x(3x-2)\,dx = 3x^2 - 2x = x^3 - x^2 + C$$

As with the derivative example above, both of these equalities are incorrect.

The minute you drop the integral sign you are saying that you've done the integration. So the first equality is saying that the value of the integral is $3x^2 - 2x$, when in reality all you're doing is simplifying the function before you integrate it.

Likewise, the last equality says that the two functions, $3x^2 - 2x$ and $x^3 - x^2 + C$ are equal, when they are very definitely not!

Here is the correct way to work out this problem:

$$\begin{aligned}\int x(3x-2)\,dx &= \int 3x^2 - 2x\,dx \\ &= x^3 - x^2 + C\end{aligned}$$

5.7.2 Dropping the dx

Another big problem in integration notation is where students drop the dx at the end of integrals. For instance:

Example 20: Dropping the dx

$$\int 3x^2 - 2x$$

The purpose of the dx in integration notation is that it tells us where the integral stops. If it's missing altogether, then the integral will be ambiguous. Here, this integral could mean a couple of different things:

$$\int 3x^2 - 2x\,dx = x^3 - x^2 + C$$
$$\int 3x^2\,dx - 2x = x^3 - 2x + C$$

Without the dx where exactly does the integral end?

The primary purpose of integration notation is to tell you that you are integrating something. And you will need to know where the integration process starts, and where it ends. To tell you where the integration starts, well, that's the role of the integral sign, $\int$; and the role of the dx is to tell you where it ends.

So the best way to think of integration notation is as a pair of brackets. Now we know that brackets always come in pairs: "(" and ")". You don't open a pair of brackets without closing it. You can think of the integration sign as an opening bracket and the dx as the closing bracket of a pair, surrounding a function that you want to integrate. You *never* have one without the other.

5.7.3 Dropped Limits

Another dropped notation error that I see on a regular basis is with definite integrals. Students tend to drop the limits of integration after the first step and do the rest of the problem with implied limits. For example:

Example 21: Dropped Limits

$$\int_1^2 x(3x-2)\,dx = \int 3x^2 - 2x\,dx = x^3 - x^2 = 8 - 4 - (1-1) = 4$$

Again, the first equality here just doesn't make sense! The answer to a *definite* integral is a *number*, while the answer to an *indefinite* integral is a *function*. When written as above you are saying the answer to the definite integral and the answer to the indefinite integral are the same, when they clearly aren't!

Likewise, the second to last equality just doesn't make sense. Here you are saying that the function, $x^3 - x^2$ is equal to $8 - 4 - (1-1) = 4$ which is just not true! Here is the correct way to work through this problem:

$$\begin{aligned} \int_1^2 x(3x-2)\,dx &= \int_1^2 3x^2 - 2x\,dx \\ &= \left[x^3 - x^2\right]_1^2 \\ &= (8-4) - (1-1) \\ &= 4 \end{aligned}$$

5.8 Loss of Notation in General

The previous three topics that I've discussed have all been examples of dropped notation errors that students first learning calculus tend to make on a regular basis. Be careful with these kinds of errors. You may know what you're trying to say, but bad notation may imply something totally different.

Remember that in many ways written mathematics is a language. If you mean to say to someone

"I'm thirsty, could you please get me a glass of water to drink."

you wouldn't drop words that you considered extraneous to the message and just say

"Thirsty, drink."

This is ambiguous and the person that you were talking to may get the idea that you are thirsty and wanted to drink something. They would definitely not get the idea that you wanted *water* to drink or that you were asking them to get it for you. Or they might interpret this as a command - that *they* had to drink something! You would know what is was you were trying to say, but just those two words would not convey your message to anyone else.

This may seem like a silly example, because you would never do something like this. You would give the whole sentence and not just two words because you are fully aware of how confusing simply saying those two words would be.

But that, however, is exactly the point of the example.

You know better than to skip important words in spoken language, so you shouldn't skip important notation (i.e. words) in writing down the language of mathematics. You may feel that they aren't important parts to the message, but they are. Anyone else reading the message you wrote down would not necessarily know that you neglected to write down those important pieces of notation and would very likely misread the message you were trying to impart.

So, be careful and use proper notation. In exams, the marker will mark the "message" you write, not the "message" you meant. The examiner can't read your mind so she won't even try to. If the "message" that she reads in marking your exam is wrong, she will mark it accordingly.

5.9 Dropped Constant of Integration

Dropping the constant of integration on indefinite integrals (the $+C$ part) is one of the biggest errors that students make in integration.

There are actually two errors that are made here. Some students just don't put it in at all, and others drop it from intermediate steps and then just tack it onto the final answer at the end.

Those that don't include it at all tend to be the students that don't remember (or never really understood) that the indefinite integrals give the most general possible function that we could differentiate to get the integrand (the function we integrated). Because we need the most general possible function, we've got to include the constant, since constants differentiate to zero.

For those that drop it from all intermediate steps and just tack it on at the end there are other issues. I suppose that the problem is that these (in fact it's probably most) students just don't see why it's important to include the constant of integration.

The first place where constants of integration play a major role is in *solving differential equations*. Here the constant of integration will show up in the middle of the problem. If it's dropped there and then just tacked back on at the end to give the final answer (or not put in at all), the answer will be *very* wrong because the function you get without dropping it will be totally different from the function you get if you do drop it!

5.10 Transforming Limits on Integrals

When you first start to integrate functions, you come across the idea of *limits* on integrals. These are normally written at the top and the bottom of the integral sign like this:

$$\int_0^2$$

and what that means in something like this

$$\int_0^2 (x+5)^2 \, dx \tag{2}$$

is that 0 and 2 are values of x. I'd bet that every book you have ever seen, and every exam, and every teacher you've ever had, write limits this way.

The problem with this is that when you start learning about one of the integral transformation techniques, called "substitution", where the idea is that you transform the integral by changing the variable from x to something else, then you open up the way for a specific type of mistake.

When you transform an integral like (2), you do it by introducing a new variable, u, say, that you hope will transform the integral from something you can't integrate, into something you can integrate.

And you do the transformation like this. Let $u = x + 5$. Then when you go through the motions, and you transform the function (the $(x+5)^2$ bit) into u-stuff, and you transform the differential (the dx bit), into u-stuff, you get

$$\int_0^2 u^2 \, du \tag{3}$$

And at this point it's really easy to fall into the trap of thinking that you've done all the transforming you need to do. As I very often used to do when I did A-Level Maths.

But those limits are *values of x, not u!* So we have to transform the limits from x-stuff to u-stuff as well! And when you write your limits as in (3) (as everyone does), then you can easily fall into the trap of not realising that you have to transform the limits of your integral.

However, if you wrote your limits like this

$$\int_{x=0}^{x=2} (x+5)^2 \, dx$$

(showing very clearly that 0 and 2 are both values of x) then after transforming the function and the differential, you get

$$\int_{x=0}^{x=2} u^2 \, du$$

and it's really clear that you have not transformed your limits yet. You have vastly reduced the possibility of making the I-forgot-to-transform-my-limits-again mistake.

6 Common Errors

6.1 Read the Questions!

This is probably one of the biggest mistakes that students make. You've got to read questions *very carefully*. Make sure you understand what you are being asked to do *before* you start working on a problem.

If you are doing homework, you might run with the assumption : "It's in Chapter X so they must want me to ...." But in many cases you simply can't assume that. Do not just skim through the question or read the first few words and assume you know the rest.

Questions will often contain information (sometimes in a very subtle way) pertaining to the steps that your examiner wants to see and the form the final answer must be in. Also, many mathematical problems can proceed in several ways depending on one or two words in the question. If you miss those one or two key words, you may end up going down the wrong path and getting the problem completely wrong.

Not reading the instructions is possibly the biggest source of dropped marks in exams.

6.2 Answer the Questions That Have Been Set!

I had a student once (let's call him "Tarquin" to protect the guilty) who needed an A in Maths to get into the university of his choice to do the subject of his choice.

He was having a few problems with some of the C4 topics (he needed an A in C4), so we spent quite a lot of time going over those, finding strategies to answer the questions that worked for him. One such topic was that of binomial expansions. I showed him a style that I use to minimise the ways it's possible to go wrong when answering this type of question (see Section 11); we did quite a lot of practice answering binomial expansion questions; we did lots of past papers. It was all going tickety-boo.

Tarquin came out the exam and immediately texted me. "I've nailed it!" he said. He couldn't have been happier.

Come results day: D.

Neither of us could believe it, so Tarquin asked to get his answer paper back to see what had happened. When it arrived the first thing we did was to go to the binomial expansion question. The question was: find the binomial expansion of

$$f(x) = \frac{1}{\sqrt{4+x}} \qquad |x| < 4$$

So we looked at his answer, the first line of which read:

$$f(x) = \frac{1}{\sqrt{4-x}}$$

So he'd failed to copy the function from the question to be the first line of his answer. The rest of his working was a perfect answer to the *wrong* question: he'd correctly expanded the *wrong* function. This question was worth 6 marks, and he got (from memory) *one* of them. For a perfect answer.

And do you know what? He'd done exactly the same thing - providing a perfect answer to the wrong question - *three* more times on that C4 paper.

That cost Tarquin a year, as he had to wait until the next round of exams to retake C4. He got his A in C4 the following January, and then had to wait until the next September to go off to university.

If the question asks you to solve this pair of simultaneous equations

$$x + y = 2$$
$$x^2 + 2y = 12$$

then there's not much point spending ten minutes solving these

$$x + y = 3$$
$$x^2 + 2y = 12$$

is there? Even if your solution is flawless, you're not going to get many marks.

Don't do a Tarquin. As soon as you have written down the thing to start you off on a question, *check it* to make sure you've written it down correctly *before* you start doing any work. You never know, it might save you a year.

6.3 Remember Restrictions on Formulas

This is an error that is often compounded by instructors (me included, I must admit) who don't mention the restrictions that apply to formulas. In some cases the instructors forget the restrictions, in others they seem to have the idea that the restrictions are so obvious that they don't need to be mentioned, and in other cases the instructors just don't want to be bothered with explaining the restrictions so they don't talk about them.

For instance, in learning about surds you should have run across the following formula:

$$\sqrt{ab} \equiv \sqrt{a}\sqrt{b}$$

The problem is that there is a restriction on this formula, and many instructors don't bother to mention it and so students aren't always aware of it. Even if instructors do mention the restriction on this formula many students forget it as they are rarely faced with a case where the formula doesn't work.

Take a look at the following example to see what happens when the restriction is violated (I'll give the restriction at the end of the example). Warning: if you haven't done complex numbers, then you can skip this example!

Example 22: Proving $1 = -1$

$\sqrt{1} = \sqrt{1}$	1 : This is certainly a true statement.
$\sqrt{(1)(1)} = \sqrt{(-1)(-1)}$	2 : Since $1 \equiv (1)(1)$ and $1 \equiv (-1)(-1)$.
$\sqrt{1}\sqrt{1} = \sqrt{-1}\sqrt{-1}$	3 : Using the $\sqrt{ab} \equiv \sqrt{a}\sqrt{b}$ property on both roots.
$(1)(1) = (i)(i)$	4 : Since $\sqrt{-1} \equiv i$
$1 = i^2$	5 : Just a little simplification.
$1 = -1$	6 : Since $i^2 \equiv -1$

So clearly we've got a problem here as we should be well aware that $1 \neq -1$! The problem arose in going from line 2 to line 3.

The snag here is that the formula $\sqrt{ab} \equiv \sqrt{a}\sqrt{b}$ has the important restriction on it that *a and b can't both be negative*. It's okay if one or the other is negative, but they can't *both* be negative!

There is also an example from calculus of this kind of ignoring-the-restriction problem.

Example 23: A Restriction on the Polynomial Differentiation Rule

The first formula that you learn in differentiation is

$$\frac{d}{dx}\left(x^n\right) \equiv nx^{n-1}$$

This is where most instructors leave it, despite the fact that there is a fairly important restriction that needs to be mentioned as well. I suspect most instructors are so used to using the formula that they just implicitly feel that everyone knows the restriction and so don't have to mention it.

In order to use this formula *n must be a fixed constant*! In other words you can't use the formula to find the derivative of x^x, for example, since in that case the exponent is not a fixed constant, it's a variable.

If you did try to use the above rule to find the derivative of x^x you would arrive at

$$\frac{d}{dx}\left(x^x\right) \equiv x \cdot x^{x-1} \equiv x^x$$

whereas the correct derivative is

$$\frac{d}{dx}\left(x^x\right) \equiv x^x\left[1 + \ln(x)\right]$$

So, you can see that what we got by incorrectly using the formula is not even close to the correct answer.

6.4 Does Your Answer Make Sense?

When you've finished working on a problem, make sure that your answer is reasonable. If it's not reasonable then you've probably made a mistake so go back and try to find it.

For example, in problems involving arithmetic series, we occasionally encounter problems where we invest a certain amount of money in an account that earns interest at a specific rate over a period of time (usually for a specific number of years/months/days depending on the problem). These are called *compound interest* problems.

If you are earning interest then the amount of money should *grow*, so if you end up with *less* than you started with you've made a mistake. Likewise, if you only invest £2000 for a couple of years at a small interest rate you shouldn't have a couple of billion pounds in the account two years later.

Often the mistake that gives an obviously incorrect answer is an easy one to find. So, check your answers and make sure that they make sense!

Here's another simple example. If you were asked to find the equation of this line (given some more information than this, obviously!),

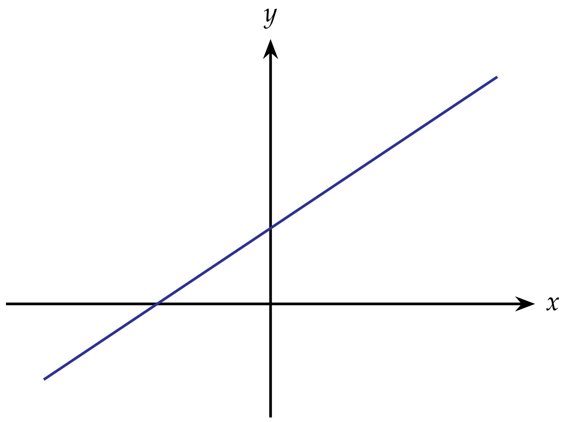

Figure 2: Finding the Equation of a Straight Line

and you get

$$y = -\frac{1}{2}x + 2$$

then that can't possibly be right, can it? Why not? Well, lines like the one above have *positive* gradients. But the equation you've found has a *negative* gradient. So it can't possibly be right, and you must have mad a mistook. Go back and find it. You can continually monitor your work for this kind of thing. You can continually...

6.5 Check Your Work

This is *so* important! *Check your work*! You will often catch simple mistakes by going back over your work, or by monitoring your work as you go.

The best way to do this is to put your work away then come back and redo all the problems and compare your new answers to those you got the first time round. This is time consuming and so can't always be done (you wouldn't have time to do this in an exam, for instance), but it's the best way to check your work.

If you don't have that kind of time available to you, then at least read through your work. You won't catch all the mistakes this way, but you might catch some of the more glaring ones.

You might also want to check your answers with those of other students. Some instructors frown on this and want you to do all your work individually, but if your instructor doesn't mind this, it's a nice way to catch mistakes.

If you are going to rely on reading through your work, rather than redoing it, then your work needs to be easily readable. Because of this, you should develop a style when you do mathematics of including in your work *what it is that you are doing at each step*. That way it will be much easier to check than a sheet of A4 full of algebra.

As a simple example, check this out:

Example 24: Solving $x^2 - 7x + 12 = 0$

If you were to solve this problem by just doing the algebra it might look something like this:

$$x^2 - 7x + 12 = 0$$
$$(x-3)(x-4) = 0$$
$$x = 3 \text{ or } x = 4$$

But, if you just add a little bit of explanation as you go:

Starting with

$$x^2 - 7x + 12 = 0$$

factorise:

$$(x-3)(x-4) = 0$$

so either:

$$x - 3 = 0 \text{ or } x - 4 = 0$$

and so:

$$x = 3 \text{ or } x = 4$$

then the whole thing is much easier to read, and hence check. Not only for you, but for an examiner, too!

If you don't like writing lots of words, you could use a bit of shorthand:

Example 25: Solving $3x^2 - 2 = 10$

$$3x^2 - 2 = 10$$

$+2$

$$3x^2 = 12$$

$\div 3$

$$x^2 = 4$$

$\sqrt{}$

$$x = \pm\sqrt{4} \qquad \text{(see Section 2.6!)}$$

esr

$$x = 2 \text{ or } x = -2$$

where my "esr" means "evaluate square root". Doing this kind of thing makes your work *much* easier to check later, as you don't have to spend time working out what you did to get from each line to the next. It's there for you to read. Now you just need to check it.

I would highly recommend that you do this kind of thing. Not just as you practice, but also for homework and exams. And another thing: as you develop this style of explaining what you are doing as you go, also follow the maxim of *only do one thing at a time*. That is, in any single step in an algebraic argument, only do *one* thing (to both sides).

This idea is an example of a big general idea. In an exam, don't do something in your head. Write it down. You can't get marks for something you did in your head. You only get marks for the things you write down on the paper and hand in. And if you've got a head that's anything like mine, then if you do something in your head, you're quite likely to get it wrong!

6.6 Guilt by Association

This is a very poor title. But if I give you an example of the kind of errors that I'm talking about, I'm hopeful that you will see what I mean.

Too often students make the following logic errors. Since the following formula is true

$$\sqrt{ab} \equiv \sqrt{a}\sqrt{b} \qquad \text{where } a \text{ and } b \text{ can't both be negative(!)}$$

they reason that there must be a similar formula for $\sqrt{a+b}$. In other words, if the formula works for one algebraic operation (i.e. addition, subtraction, division, and/or multiplication[2]) it must work for all. The problem is that this usually isn't true! In this case

$$\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$$

Similarly, in calculus students make the mistake that because

$$(f+g)' \equiv f' + g'$$

where derivatives are being *added*, then the same must be true for a *product* of functions. Again, however, it doesn't work that way!

$$(fg)' \neq f' \cdot g'$$

So, don't try to extend formulas that work for certain algebraic operations, to all algebraic operations. If you were given a formula for a certain algebraic operation, but not others, there was very probably a very good reason for that. In all likelihood it only works for those operations for which you were given the formula!

We've seen all this before...check out Sections 2.4, and 5.1.1.

6.7 Bad Notation

Bad notation always sets me on edge when I see it. First, I see the following all too often,

$$2 + x - 6x = 2 + -5x$$

The $+-5x$ is really poor! It combines into a negative *so write it like that*! Here's the correct way:

$$2 + x - 6x = 2 - 5x$$

Next, one (the number) times something is just the something, so there is no reason to continue to write the one. For instance, don't write $2 + 7x - 6x$ as $2 + 1x$! The coefficient of one is not needed here since $1x \equiv x$!

This same thing holds for a power of one. Anything to the first power is the anything so never have 1 as a power!

6.8 Rounding Errors

For some reason some students seem to develop the attitude that everything must be rounded as much as possible, and as often as possible. This has gone so far that I've actually had students who refused to work with decimals! Every answer was rounded to the nearest integer, regardless of how wrong that made the answer.

There are simply some problems where rounding too much can get you into trouble and seriously change the answer. The best example of this is compound interest problems. Here's a quick example.

It turns out that if you invest P pounds at an interest rate of $r\%$ that is compounded m times per year, then after t years you will have A pounds where

$$A = P\left(1 + \frac{r}{100m}\right)^{mt}$$

[2]Should I say: $\frac{\text{and}}{\text{or}}$ multiplication??

Example 26: A Compound Interest Problem

So, let's invest £10,000 at an interest rate of 6.5% compounded monthly for 15 years. So,

$$
\begin{aligned}
P &= 10,000 \\
r &= 6.5 \\
m &= 12 \text{ , and} \\
t &= 15
\end{aligned}
$$

So, here's what we'll have after 15 years:

$$
\begin{aligned}
A &= 10,000\left(1+\frac{6.5}{1200}\right)^{12\times 15} \\
&= 10,000\,(1.005416667)^{180} \\
&= 10,000 \cdot 2.644200977 \\
&= 26,442.00977 \\
&= 26,442.01 \text{ (to 2 d.p.)}
\end{aligned}
$$

So, after 15 years we will have £26,442.01. You will notice that I didn't round until the very last step and that was only because we were working with money which usually only has two decimal places.

Next is the same example, showing you just how much difference rounding too much can make. This time, at each step I'll round each answer to three decimal places, apart from the last step where I round to two decimal places (because it's money):

$$
\begin{aligned}
A &= 10,000\left(1+\frac{6.5}{1200}\right)^{12\times 15} \\
&= 10,000\,(1.005)^{180} \\
&= 10,000 \times 2.454 \\
&= 24,540.00 \text{ (to 2 d.p.)}
\end{aligned}
$$

Getting the idea? This is getting on for being £1000 out!

The general rule is: if you've calculated a number, and you are going to use that number in another calculation later in the question, then use *the most accurate value of the number* you have in any subsequent calculation.

Rounded values of numbers are only for display purposes to convey the number to the marker of your work.

Part II

...And One Way to Prevent Them

7 Introduction

For me, the best way of preventing errors is to have a *style* for answering particular kinds of problems. If you have a style for answering a particular kind of problem, then every time you do one of these questions, it looks familiar. The same kinds of thing turn up. Because of the familiarity, you can often quickly spot where you have slipped up.

The following are several examples of what I mean.

8 Integration by Substitution

See [Smith(2016b)].

9 Integration by Parts

See [Smith(2016a)].

10 The Trapezium Rule

10.1 A Question

Use the trapezium rule with 5 intervals to estimate the value of

$$\int_{x=0}^{x=1} \tfrac{1}{10}x^2 + 1 \, dx$$

Draw the graph of $y = \frac{1}{10}x^2 + 1$, and explain why the trapezium rule gives an overestimate of the true value of the integral.

10.2 My Answer

Whenever I do a Trapezium Rule question, I *always* draw a picture of the situation. Figure 3 is my picture for this question.

In this question, it specifically asks me to draw the thing, but I would have done it anyway. One of my basic principles when I'm doing any kind of question is : "How can I draw a picture of this question?".

There are 5 intervals, so each interval will have width h, of $\dfrac{1-0}{5} = 0.2$.

The next thing I do is to work out my own formula for the area I want. As the area of a trapezium of width h, and vertical sides a and b, is $\frac{1}{2}h(a+b)$, then

$$\begin{aligned}
A_1 &= \tfrac{1}{2}h(y_0 + y_1)\\
A_2 &= \tfrac{1}{2}h(y_1 + y_2)\\
A_3 &= \tfrac{1}{2}h(y_2 + y_3)\\
A_4 &= \tfrac{1}{2}h(y_3 + y_4)\\
A_5 &= \tfrac{1}{2}h(y_4 + y_5)
\end{aligned}$$

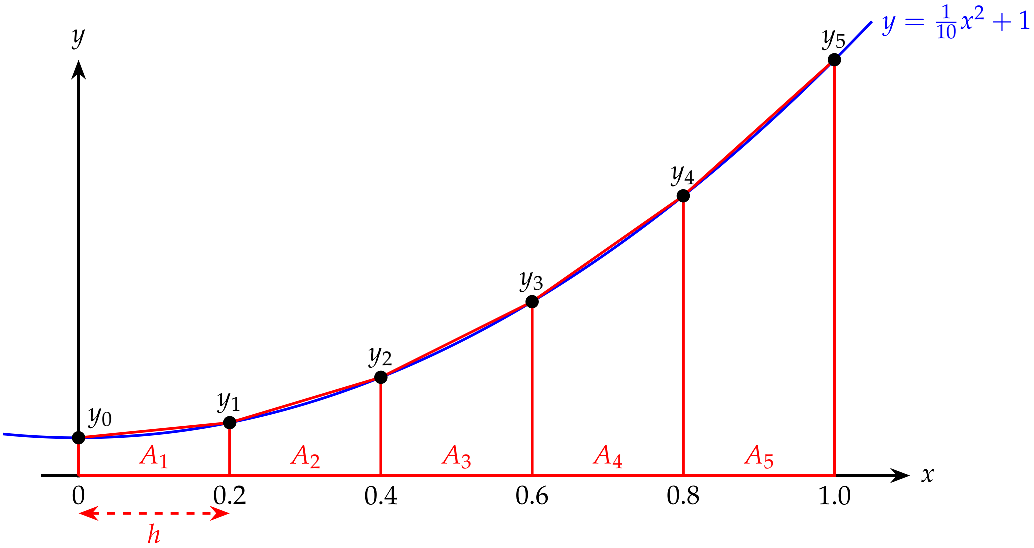

Figure 3: Trapezium Rule Question

and if I add all these up I get

$$A = A_1 + A_2 + A_3 + A_4 + A_5 = \tfrac{1}{2}h\Big[(y_0 + y_1) + (y_1 + y_2) + (y_2 + y_3) + (y_3 + y_4) + (y_4 + y_5)\Big]$$
$$= \tfrac{1}{2}h\Big[y_0 + 2y_1 + 2y_2 + 2y_3 + 2y_4 + y_5\Big]$$

There's my formula. Notice that I *don't* use the formula given to me in the formula booklet. I work out my own, and tailor it to the specific question I've got.

Next I have to work out all the y-values:

i	x_i	y_i
0	0.0	1.000
1	0.2	1.004
2	0.4	1.016
3	0.6	1.036
4	0.8	1.064
5	1.0	1.100

Table 1: y-values for $y = \frac{1}{10}x^2 + 1$

Now I've done that, I can just plug the y-values into my formula:

$$A = \tfrac{1}{2} \times 0.2 \times \Big[1.000 + 2 \times 1.004 + 2 \times 1.016 + 2 \times 1.036 + 2 \times 1.064 + 1.100\Big]$$
$$= 1.034$$

Finally, I go through a *checking* stage.

So: could this answer of 1.034 possibly be correct? How can I tell? Well, if you look at the y-values in Table 1, you will see that they're all very similar. They're all about 1. So an estimate for the area under this curve would be 1×1 (I've simplified the area to a rectangle, which has a height of 1 and a width of 1). And that gives me an approximate area of 1. So my answer is looking good.

Another general strategy: once you've got an answer to a question, ask yourself: "Is this answer reasonable?", and "How can I check this?". Thinking of ways to check what you're doing as you go is a sure-fire recipe for finding mistakes early.

And that's a good thing, isn't it?

The final part of this question is about whether my answer will be an under-estimate or an over-estimate.

Well, from Figure 3, I hope you can (just!) see that the area of each trapezium is slightly larger than the corresponding area under the curve. So our estimate will be an overestimate.

10.3 Trapezium Rule Questions : My Style

Draw a picture of the area to be estimated, divided up into trapezia

↓

Work out my own formula

↓

Find all the y-values

↓

Plug the y-values into my formula

↓

Find a way of estimating the answer!

Figure 4: The Trapezium Rule : My Style

So - why do I go to all this palaver? Why don't I just use the formula given in the formula book?

Well, I could. But going through this process of drawing the pictures, working out my own formula, and plugging the y-values into it, forces me to *understand* what I'm doing.

And I find that when I fully understand something, it is often obvious when I've made a mistake.

And don't forget the checking step: that will pick up mistakes, too!

11 Binomial Expansions

11.1 A Question

Expand $f(x)$, where

$$f(x) = (1-2x)^n, \quad n \in \mathbb{N}$$

as far as the term in x^3.

11.2 My Answer

The way I expand binomials is like this:

$$\begin{aligned}(1-2x)^n = 1 \\ &+ \frac{n}{1} \cdot (-2x)^1 \\ &+ \frac{n}{1} \cdot \frac{n-1}{2} \cdot (-2x)^2 \\ &+ \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot (-2x)^3 \\ &+ \ldots\end{aligned}$$

Again, I'm not using the formula in the formula book, but what I do will result in the same thing as the booklet formula. However, by writing the formula *vertically*, rather than horizontally (as in the formula booklet), I get a visual pattern for how the formula is developing. And so by looking for places where the pattern is faulty, I can spot mistakes.

I obtain this formula by:

- I make sure the first term in the binomial is 1. If it is, then the first line of the formula will be 1.
- The next line will be $\frac{n}{1}$ times whatever comes after the 1 in the binomial (I always put that bit in brackets. Note that this *includes the sign*) raised to the power 1.
- On each successive line, the coefficient has an extra fraction, where the number on the top goes down by 1 and the number on the bottom goes up by 1. And the power of the bit in brackets goes up by 1.

This way, you develop a pattern, which you can check (a common theme when you are trying to prevent mistakes - check regularly, wherever possible) before you go to the next step of simplifying the expansion.

Once we're happy, we can simplify:

$$(1-2x)^n = 1 - 2nx + 2n(n-1)x^2 - \tfrac{4}{3}n(n-1)(n-2)x^3 + ...$$

Putting the thing after the 1 in the binomial in brackets eliminates the problem of forgetting to take powers of the minus sign as you go...

11.3 Binomial Expansions : My Style

Make the number that comes first in the binomial 1. Let the binomial we now have be $(1+a)^n$

↓

The first line of the formula is then 1

↓

The next line of the formula will be $\frac{n}{1} \cdot (a)^1$

↓

The next line of the formula will be $\frac{n}{1} \cdot \frac{n-1}{2} \cdot (a)^2$

↓

The next line of the formula will be $\frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot (a)^3$, etc

Figure 5: Binomial Expansions : My Style

12 Dynamics Problems in Mechanics (+ *SUVAT*)

12.1 A Question

Father Christmas has parked his sleigh (weight 5000N) on a roof inclined at an angle of 30° to the horizontal. Because of the snow on the roof, the coefficient of friction between the sleigh and the roof is only 0.1.

As soon as the sleigh is parked, it starts to slide down the roof. If the distance from the initial position of the sleigh to the edge of the roof is 5m, find out how long Santa has to put all the presents down the chimney, and get back onto the sleigh before it falls off the roof.

12.2 My Answer

The first thing to do is to draw all the real forces on the sleigh. They will be:

- it's weight,
- the normal reaction,
- friction.

See Figure 6.

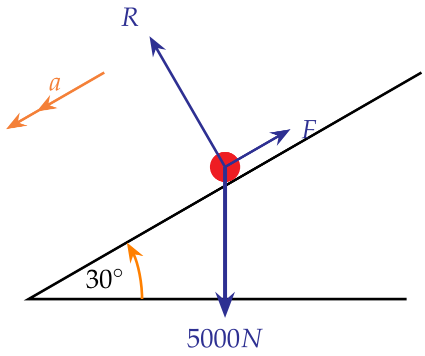

Figure 6: The Real Forces on the Sleigh

Next we pick two purpledicular directions:

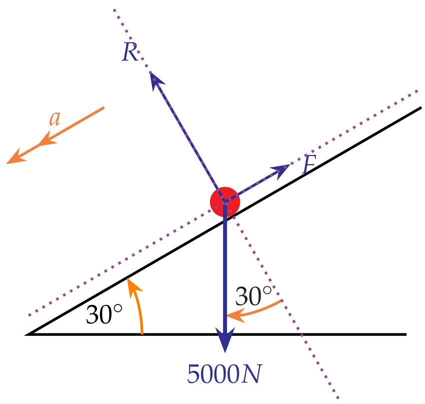

Figure 7: Picking the Two Purpledicular Directions

We now have to get all our forces in these directions. We do that by finding the components of the forces that are not in these directions. The only one we need to find components for is the weight:

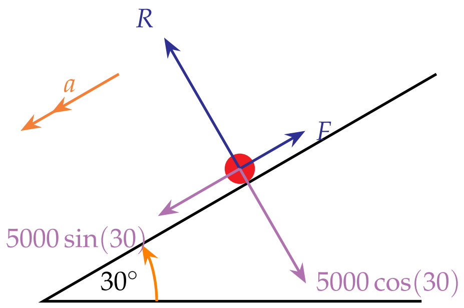

Figure 8: Finding Components

Now we can find the equations that apply to our sleigh. These will be:

normal to the plane: equilibrium	$R = 5000\cos(30^\circ)$
parallel to the plane: Newton II	$5000\sin(30^\circ) - F = \dfrac{5000}{g}a$
friction	$F = 0.1R$

From these equations we can see that

$$F = 0.1 \cdot 5000\cos(30^\circ) = 500\cos(30^\circ)$$

and so

$$5000 \sin(30^\circ) - 500 \cos(30^\circ) = \frac{5000}{g} a$$

Solving for a gives

$$\left[5000 \sin(30^\circ) - 500 \cos(30^\circ)\right] \cdot \frac{g}{5000} = a \approx 4.06\ ms^{-2}$$

Right. So if the sleigh starts off 5 m from the edge of the roof, and it takes a time t to slide there, accelerating at a rate of 4.06 ms^{-2}, then we have a *SUVAT* problem! And guess what? I have a *style* for doing *SUVAT* problems!

In any *SUVAT* problem, there are five variables, and five equations. You can associate each equation with one of the variables, because each of the equations is missing one of the variables. So for example, I associate the equation $v = u + at$ with the variable s because that variable is missing from that equation.

Each equation has four different variables in it. If one of those variables is the thing we want, then we must know the values of the other three variables to find it. That means we need *three* things in the "Wot We Know" column of my *SUVAT* table (see Table 2).

	Wot We Know	Wot We Want	Equation
s			$v = u + at$
u			$s = vt - \frac{1}{2}at^2$
v			$s = ut + \frac{1}{2}at^2$
a			$s = \frac{1}{2}(u + v)t$
t			$v^2 = u^2 + 2as$

Table 2: Initial *SUVAT* for the Sleigh Sliding Off the Roof

In our problem, we know u, the initial velocity (which will be 0), a and s. We neither know nor want v, and we need to find t. Let's update our table.

	Wot We Know	Wot We Want	Equation
s	5 m		$v = u + at$
u	0 ms^{-1}		$s = vt - \frac{1}{2}at^2$
v			$s = ut + \frac{1}{2}at^2$
a	4.06 ms^{-2}		$s = \frac{1}{2}(u + v)t$
t		✓	$v^2 = u^2 + 2as$

Table 3: Updated *SUVAT* for the Sleigh Sliding Off the Roof

Which equation do we use? Easy: as we neither know nor want v, we use the equation without v in it. See Table 4

	Wot We Know	Wot We Want	Equation
s	5 m		$v = u + at$
u	0 ms^{-1}		$s = vt - \frac{1}{2}at^2$
v			$s = ut + \frac{1}{2}at^2$
a	4.06 ms^{-2}		$s = \frac{1}{2}(u + v)t$
t		✓	$v^2 = u^2 + 2as$

Table 4: Deciding Which Equation to Use

So,

$$s = ut + \tfrac{1}{2}at^2$$

Since $u = 0$ we have

$$s = \tfrac{1}{2}at^2$$

and solving for t gives

$$\begin{aligned} t &= \sqrt{\frac{2s}{a}} \\ &= \sqrt{\frac{2 \times 5}{4.06}} \\ &= 1.57s \end{aligned}$$

Santa's going to have to get a move on!

12.3 Dynamics Problems in Mechanics : My Style

Draw a picture of all the real forces acting on the object

↓

Pick two purpledicular directions

↓

Find the components of the forces that aren't in these directions

↓

Obtain the equations that apply to the different directions, and friction

↓

Solve the equations

Figure 9: Dynamics Problems in Mechanics : My Style

12.4 *SUVAT* Problems : My Style

Draw up the *SUVAT* table

↓

Find *three* things to put in the "Wot We Know" column

↓

Decide "Wot We Want"

↓

Select the appropriate equation

↓

Solve it

Figure 10: *SUVAT* Problems : My Style

Part III

Appendices

A Inconsistent Mathematical Notation

This is a bit of fun. Or is it?

When you start learning algebra, you are told that when you see two symbols next to each other, like this:

$$ab$$

then that means a times b.

OK, you might think. Got that. So when I see something like this:

$$34$$

then clearly that means 12, because $3 \times 4 = 12$. Oh, but then $3 \times 4 = 12 = 1 \times 2 = 2$. Yep. Got it. And when I see something like this:

$$6\tfrac{1}{2}$$

then that means 3, because $6 \times \frac{1}{2} = 3$. Yep. Really got it now.

No wonder that children struggling with some of these ideas in primary school get so confused. And because there seems to be a different rule in every different situation, they give up trying to understand it, and just try to learn it by rote.

By the time they've got to the end of primary school, they have been completely turned off mathematics. And it's no fault of their own. They have been lost because the notation used in mathematics is ambiguous at best, rubbish at worst, and their teacher (who is probably not a mathematical specialist), can't understand the child's problem, and so can't fix it.

So...is this just a bit of fun?

And of course, this kind of thing doesn't stop at primary school. What do these things mean:

$$x^{-1} \qquad\qquad f^{-1}$$

The answer of course is *it depends on what x and f are*. If x is a number then

$$x^{-1} \equiv \frac{1}{x}$$

But if f is a function then

$$f^{-1} \equiv \text{the inverse function of } f$$

Admittedly, we will usually know whether we are dealing with a number or a function, but how is that serious, unambiguous mathematical notation? Wouldn't it be better if there was clearly distinguishable notation for each different mathematical idea?

B Why is $\int \frac{1}{x}\,dx \equiv \ln(|x|)$ (With a Modulus)?

Here's a picture of part of the function $y = \frac{1}{x}$, together with a couple of shaded areas.

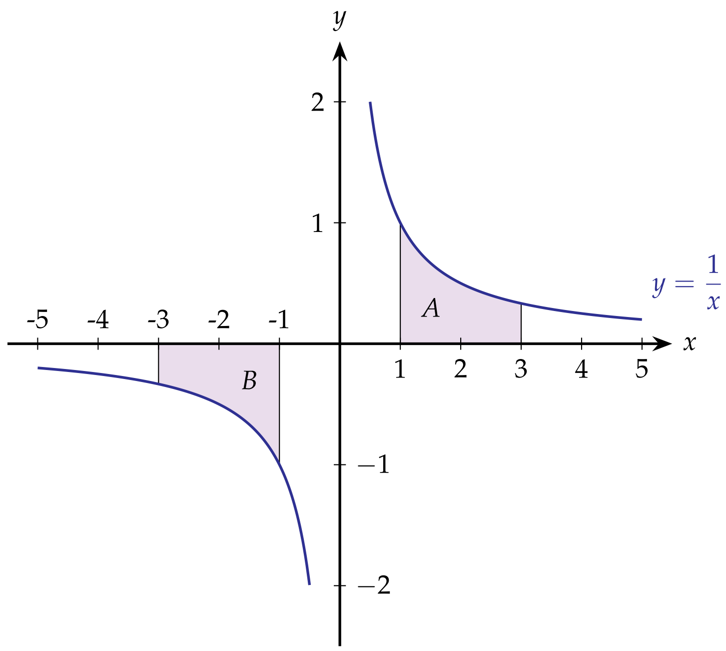

Figure 11: Areas Under $y = \frac{1}{x}$

I've drawn this graph because the relationship between the function $\frac{1}{x}$ and $\ln(x)$ is connected to finding areas under the graph of $\frac{1}{x}$. Notice that I'm ignoring the modulus signs for now.

Area A on Figure 11 is found by doing the following calculation:

$$\text{Area A} = \int_{x=1}^{x=3} \frac{1}{x}\,dx$$

Carrying on the calculation, we find that

$$\begin{aligned}\text{Area A} &= \Big[\ln(x)\Big]_{x=1}^{x=3} \\ &= \ln(3) - \ln(1) \\ &= \ln(3)\end{aligned}$$

Remember that when you integrate the function $\frac{1}{x}$, you get the function $\ln(x)$.

So now let's go through the same process, this time trying to find the area B.

$$\begin{aligned}\text{Area B} &= \int_{x=-3}^{x=-1} \frac{1}{x}\,dx \\ &= \Big[\ln(x)\Big]_{x=-3}^{x=-1} \\ &= \ln(-1) - \ln(-3)\end{aligned}$$

Now at this point we have a problem. Because log functions are not defined for negative arguments, we can't evaluate $\ln(-1) - \ln(-3)$.

But it's very clear from Figure 11 that the area B we are trying to find is very much defined. And in fact it going to have the same value as the area A, because of the symmetry of the $y = \frac{1}{x}$ function.

Right. Now let's consider what happens if we were to put in the modulus signs around the x in the log when we integrate the $y = \frac{1}{x}$ function, and try to find the area B again:

$$\begin{aligned}\text{Area B} &= \int_{x=-3}^{x=-1} \frac{1}{x}\,dx \\ &= \Big[\ln(|x|)\Big]_{x=-3}^{x=-1} \\ &= \ln(|-1|) - \ln(|-3|) \\ &= \ln(1) - \ln(3) \\ &= -\ln(3)\end{aligned}$$

This has the same value as area A, but the wrong sign! Oh, but hang on. Remember that when you find areas under a graph where the function is *under* the x-axis, the area comes out to be negative! Aha! So this does give us the right value for the area, and we just have to remember to change the sign to get the area positive.

The upshot of all this then is that if we did put modulus signs into the equation

$$\int \frac{1}{x}\,dx \equiv \ln(|x|) + C$$

then it will give us correct answers when we use it to find areas under the graph of $y = \frac{1}{x}$, even when the limits on the integral are negative. And that's because of the symmetry of the $y = \frac{1}{x}$ function.

One final point: the limits on the integral must both be positive, or both be negative. Can you see why, by looking at Figure 11?