Steve Smith Tuition: Maths Key Skills

Equations of Straight Lines

Contents

Prerequisites

None.

Notes

None.

Document History

Date	Version	Comments
4th December 2013	1.0	Initial creation of the document.

1 The General Equation of a Straight Line

(Almost!) Every straight line can be represented by the equation

$$y = mx + c \tag{1}$$

where m represents the *gradient* (or the steepness) of the line, and c represents what is known as the *y-intercept* (the place on the y-axis that the line goes through). Check out Figure 1.

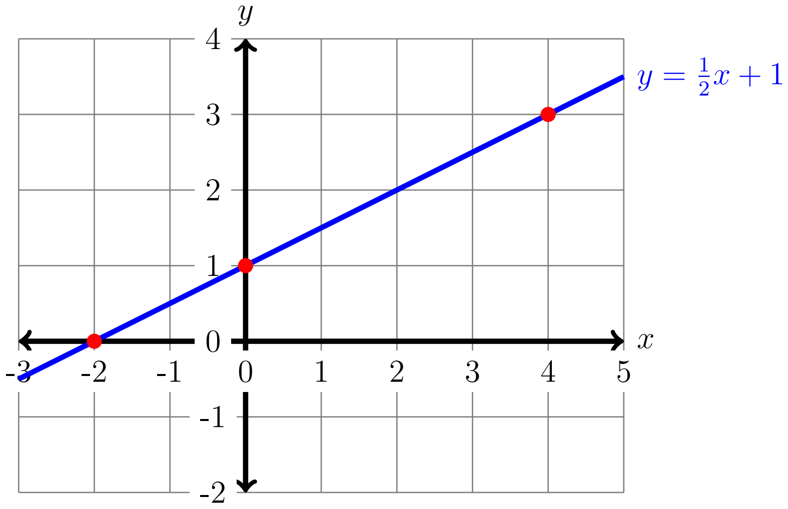

Figure 1: A graph of the straight line $y = \frac{1}{2}x + 1$

In Figure 2 I'm comparing the equation

$$y = \frac{1}{2}x + 1$$

with the general equation of a straight line, (1):

$$\begin{array}{ccccc} y & = & \frac{1}{2} & x & + & 1 \\ y & = & m & x & + & c \end{array}$$

Figure 2: Comapring the straight line $y = \frac{1}{2}x + 1$ with the general equation

and by doing this comparison, you can see that the gradient (the m) of this line must be $\frac{1}{2}$, and the y-intercept (the c) must be 1. And that tallies with the graph in Figure 1: the line passes through the point where $y = 1$ on the y-axis.

Alright. Now the topic of this set of notes is: given certain information, how do we find the equation of a line? And what sort of information would we need?

It turns out that you can find the equation of a line if you know *two* things about it. And those two things can be

- the gradient, and the coordinates of a single point, or
- the coordinates of *two* points.

2 When You Know the Gradient and a Point

Let's take an example, and see if you can understand how this works. Let's say that we were given the following information about a line:

- it's gradient is 2
- it goes through the point $(1, -1)$.

Figure 3 sums up this information.

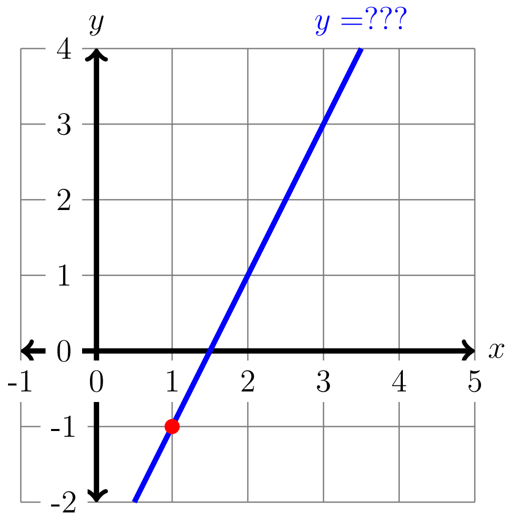

Figure 3: A graph of an unknown straight line!

The red dot shows us that the line goes through the point $(1, -1)$, and we can tell that the gradient is 2 because, for example, if you start at the point $(1, -1)$, then as you slide up the graph for each 1 you go to the right, you go 2 up.

OK, so how do we work out the equation of this line? Well, going back to the general equation of a straight line (1), we need to find the m and the c. Hang on - we know the m already! The m represents the gradient, and we know the gradient of this line! It's 2! Right, so we can write the equation of this line so far as

$$y = 2x + c$$

and all we need to do now is to find the c.

But how do we do that?

Well, there are several ways. Because this is Maths, you might expect that there's an equation to learn. And indeed there is. I'll look at that later. But my favourite method is to use a bit of reason.

Let's have a bit of a digression, and go over the idea.

A short digression...

What does the equation of a line tell you about the line? Well, it tells you the relationship between the x and y coordinates of any point on the line.

For example, looking back to the graph in Figure 1, we can see that there are several points marked with red dots on the line. Let's take one of them. Say we took the point $(4, 3)$. Now remember that the equation of this line was $y = \frac{1}{2}x + 1$. So if the point $(4, 3)$ lies on this line, then the equation should work for those values of x and y, shouldn't it? Let's try it: if we put $x = 4$ and $y = 3$ into the equation of the line,

$$3 = \tfrac{1}{2} \times 4 + 1$$

and yes! It works!

Let's try another point, say $(0, 1)$:

$$1 = \tfrac{1}{2} \times 0 + 1$$

and that works too!

But if we were to try a point that doesn't lie on the line, $(5, 2)$, say, then

$$2 \neq \tfrac{1}{2} \times 5 + 1$$

2.1 Method 1: A Bit of Logic

So my big idea is to use the equation of the line to our advantage! Here's my method. Remember that so far, in figuring out the equation of our unknown line, we have got as far as

$$y = 2x + c \tag{2}$$

and we're trying to find the c. But the thing is, we know one of the points on the line! We were told that $(1, -1)$ lies on this line! So the equation (2) must work for this point! In other words, if we insert $x = 1$ and $y = -1$ into equation (2), the equation should work! Let's do it:

$$-1 = 2 \times 1 + c$$

and this enables us to find c: first simplify the right-hand side:

$$-1 = 2 + c$$

and then subtract 2 from each side:

$$-3 = c$$

and we have found our c. So the equation for our line is

$$y = 2x - 3$$

2.2 Test 1

Find the equation of a line where:

- it's gradient is -1
- it goes through the point $(3, 3)$.

Figure 4 sums up this information.

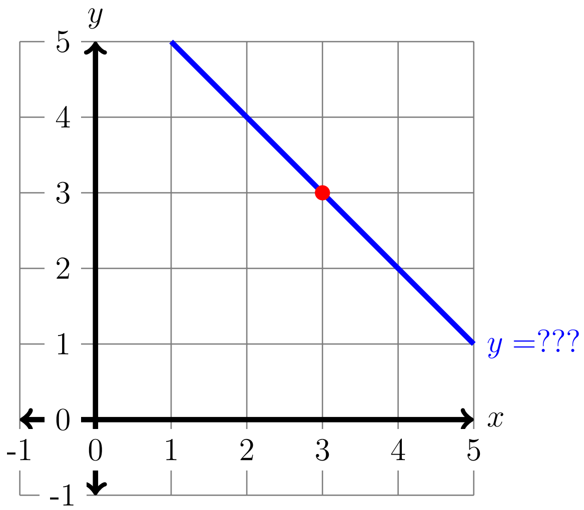

Figure 4: A graph of an unknown straight line!

2.3 Test 1 Answer

The general equation of a straight line is

$$y = mx + c$$

and we're trying to find the m and the c. But we know the gradient (the m) is -1, so we can put that into our equation:

$$y = -x + c$$

And we also know one of the points on the line! We were told that $(3, 3)$ lies on this line! So the above equation must work for this point! In other words, if we insert $x = 3$ and $y = 3$ into this equation, the equation should work! Let's do it:

$$3 = -3 + c$$

and this enables us to find c: add 3 to each side:

$$6 = c$$

and we have found our c. So the equation for our line is

$$y = -x + 6$$

2.4 Test 2

Find the equation of a line where:

- it's gradient is 3
- it goes through the point $(2, 0)$.

Figure 5 sums up this information.

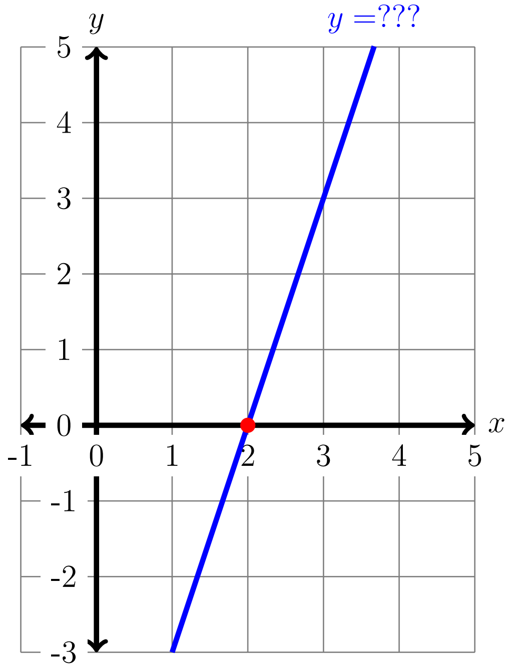

Figure 5: A graph of an unknown straight line!

2.5 Test 2 Answer

The general equation of a straight line is

$$y = mx + c$$

and we're trying to find the m and the c. But we know the gradient (the m) is 3, so we can put that into our equation:

$$y = 3x + c$$

And we also know one of the points on the line! We were told that $(2, 0)$ lies on this line! So the above equation must work for this point! In other words, if we insert $x = 2$ and $y = 0$ into this equation, the equation should work! Let's do it:

$$0 = 3 \times 2 + c$$

and this enables us to find c: simplify the right-hand side:

$$0 = 6 + c$$

and subtract 6 from each side:

$$-6 = c$$

and we have found our c. So the equation for our line is

$$y = 3x - 6$$

2.6 Method 2: An Equation to Remember

I prefer my method for finding the c in the equations of straight lines because it emphasises the ideas behind what equations of lines represent. And it also means that I don't have to remember anything. I can work stuff out using my understanding of the situation.

However, if you are one of those people who prefer to remember formulas, and you don't care whether you understand the thing or not, here's the formula to learn to find the equation of a straight line if you have the gradient, and you know the coordinates of a point on the line:

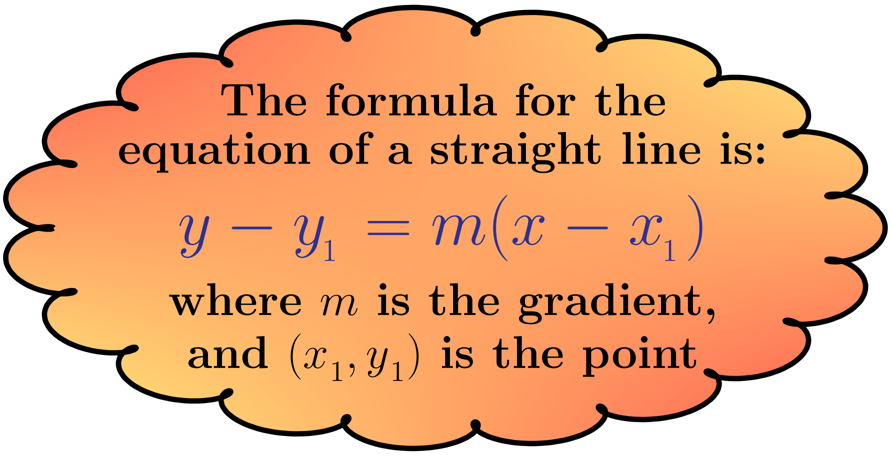

Figure 6: A formula for the equation of a straight line

Let's see how Test 1 and Test 2 can be answered using this formula.

2.7 Test 1 Answer Reprise

Well, if you know that the gradient of the line you want is -1, and it goes through the point $(3, 3)$, we just shove all this stuff into the equation:

$$y - y_1 = m(x - x_1)$$

So, putting the numbers in:

$$y - 3 = -1(x - 3)$$

Now, multiplying out the brackets on the right-hand side we get:

$$y - 3 = -x + 3$$

and then adding 3 to both sides we get

$$y = -x + 6$$

2.8 Test 2 Answer Reprise

Well, if you know that the gradient of the line you want is 3, and it goes through the point $(2, 0)$, we just shove all this stuff into the equation:

$$y - y_1 = m(x - x_1)$$

So, putting the numbers in:

$$y - 0 = 3(x - 2)$$

Now, multiplying out the brackets on the right-hand side we get:

$$y - 0 = 3x - 6$$

and simplifying the left-hand side we get

$$y = 3x - 6$$

3 When You Know Two Points

Now if you don't know the gradient, but instead you know two points on the line, then you find the gradient using the two points. Let's say that you knew that the line goes through the points $(0, 1)$ and $(4, 3)$. See Figure 7.

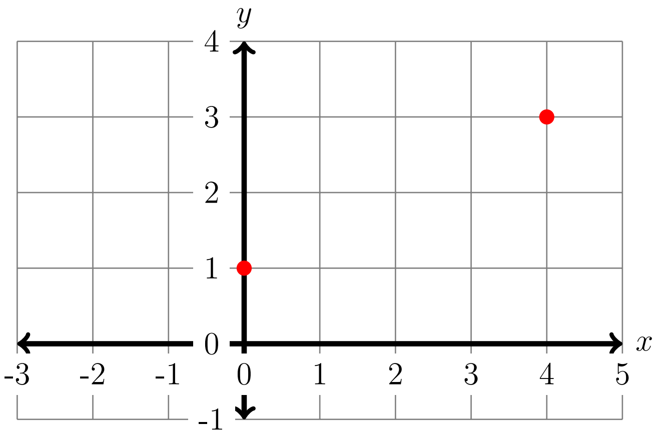

Figure 7: Two points can define a line

To find the gradient of the line joining these points, you draw a triangle underneath the points, as shown in Figure 8, and then you find the lengths of the horizontal and vertical sides. They are shown

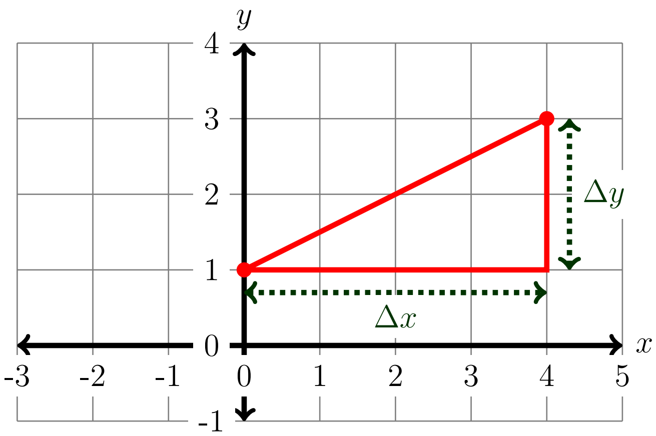

Figure 8: How to find the gradient

in Figure 8 by the distances Δx and Δy respectively[1]. And what you do with those lengths is this:

$$\text{gradient} = \frac{\Delta y}{\Delta x}$$

And in this example, looking at Figure 8 again, $\Delta y = 2$ and $\Delta x = 4$ so

$$\text{gradient} = \frac{\Delta y}{\Delta x} = \frac{2}{4} = \frac{1}{2}$$

And of course now that we know the gradient and a point (we actually know two points, but one is enough now!) then we can use the ideas of Section 2 to work out the equation of the line.

[1]By the way, the Δ symbol is often used in maths and science to denote "the change in", as it does here. So Δy means "the change in y".

Here's another example. Let's say that you knew that a line goes through the points $(1, 2)$ and $(4, 1)$. See Figure 9.

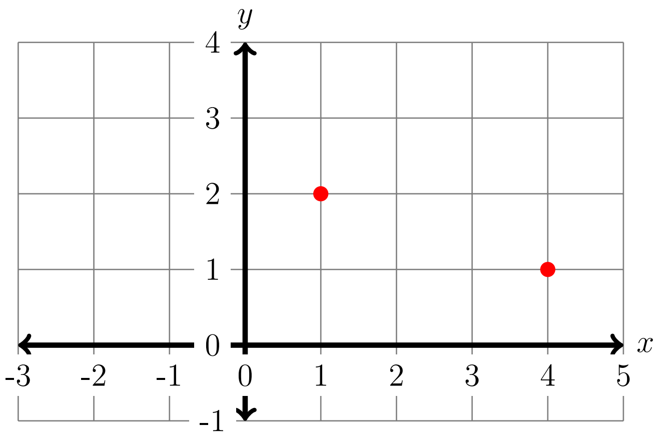

Figure 9: Two points can define a line

To find the gradient of the line joining these points, you draw a triangle underneath the points, as shown in Figure 10, and then you find the lengths of the horizontal and vertical sides. They are shown

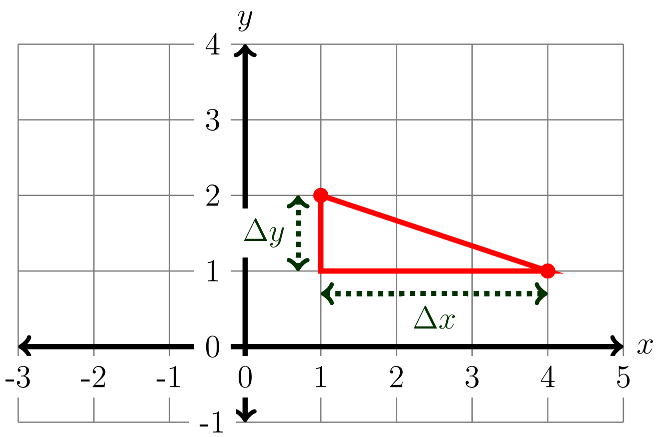

Figure 10: How to find the gradient

in Figure 10 by the distances Δx and Δy respectively. And what you do with those lengths is this:

$$\text{gradient} = \frac{\Delta y}{\Delta x}$$

And in this example, looking at Figure 10 again, $\Delta y = -1$ (because it's a downhill slope as x increases) and $\Delta x = 3$ so

$$\text{gradient} = \frac{\Delta y}{\Delta x} = \frac{-1}{3} = -\frac{1}{3}$$

And of course now that we know the gradient and a point (we actually know two points, but one is enough now!) then we can use the ideas of Section 2 to work out the equation of the line.