Steve Smith Tuition: Maths Notes

Integration by Parts, the Tabular Method, II: or "DIS is how you do more with it!"

Contents

Prerequisites

I have assumed that the reader is familiar with Integration By Parts as a tool to integrate the product of two functions.

Notes

There is a companion document to this one, *Integration by Parts, the Tabular Method, I: or "DIS is how you do it!"*, which is aimed at A-Level standard students. That outlines the DIS approach to Integration By Parts and provides lots of examples.

Document History

Date	Version	Comments
24th February 2012	1.0	Initial version of the document.
30th July 2012	2.0	Showing how DIS can produce the IBP formula.

1 Introduction

Students often find Integration By Parts (IBP) tricky. It's not that the ideas are hard, it's more that the details of the method are finicky to control, and it's easy to make a slip with a minus sign, or by integrating something that you should be differentiating, etc.

Wouldn't it be nice if there was an alternative, simple, *visual* way of doing IBP that minimised the opportunities for mistakes? Well, there is.

In this document I have called it the "DIS" method. I like that name, because when a student has trouble with IBP, I would say: "OK: check this out. DIS is how you do it!". The more usual name is the Tabular Integration By Parts method, and the literature follows that name. In Indonesia it seems to be called the Tanzalin method.

This method does not seem to be widely known. "This is unfortunate, because tabular integration by parts is not only a valuable tool for finding integrals, but can also be applied to more advanced topics including the derivations of some important theorems in analysis" ([Horowitz(1990)]).

This method is not taught in schools. It should be. Spread the word!

2 The Traditional Integration by Parts

2.1 Deriving the Formula

We start with the product rule for differentiation. If u and v are both functions of x, then the product rule states:

$$\frac{d}{dx}(uv) \ = \ u\frac{dv}{dx} \ + \ v\frac{du}{dx}$$

Now because this is an equation, and you can do anything you like to equations[1], then we could integrate both sides of this equation with respect to x:

$$\int \frac{d}{dx}(uv) \ dx \ = \ \int u\frac{dv}{dx} \ dx \ + \ \int v\frac{du}{dx} \ dx$$

But the left-hand side of this equation is the integral of the differential of something. And integration and differentiation are inverse processes, so the left-hand side simplifies to just uv:

$$uv \ = \ \int u\frac{dv}{dx} \ dx \ + \ \int v\frac{du}{dx} \ dx$$

And subtracting $\int v\frac{du}{dx} \ dx$ from both sides gives:

$$\int u\frac{dv}{dx} \ dx \ = \ uv \ - \ \int v\frac{du}{dx} \ dx \tag{1}$$

2.2 First Example

By way of an introductory example, let's integrate

$$I \ = \ \int x \sin x \ dx$$

Now from (1), we need to choose which factor of our product will be the u and which will be the $\frac{dv}{dx}$. This is a very important step in this method, because your choice is crucial. Here's the thinking. If you look at the right-hand side of (1) you see that u appears, as does $\frac{du}{dx}$. So to use the formula, we need to differentiate u. Also by inspection of the right-hand side of (1), v appears. That means we are going to have to integrate $\frac{dv}{dx}$. So we are going to have to differentiate one of the functions and integrate the other. Also, on the right-hand side of (1) we have another integral to do. And for this entire process to be useful, that integral needs to be simpler than the one we started with.

OK, so with all that in mind, let's choose $u = \sin x$ and $\frac{dv}{dx} = x$ and let's see what happens:

$$\begin{aligned} \textit{Let} \ \ u \ &= \ \sin x & \implies \frac{du}{dx} \ &= \ \cos x \\ \textit{and} \ \ \frac{dv}{dx} \ &= \ x & \implies v \ &= \ \frac{1}{2}x^2 \end{aligned}$$

[1] So long as you do the same thing to both sides!

So, plugging all this in to (1) we get:

$$\int x \sin x \; dx \;=\; \frac{1}{2}x^2 \sin x \;-\; \int \frac{1}{2}x^2 \cos x \; dx$$

which is all very well, but the problem is that the integral on the right-hand side ($\int \frac{1}{2}x^2 \cos x dx$) is now worse than the one we started with ($\int x \sin x dx$), because of the x^2 bit.

Now let's have a look at what happens when we swap the u and $\frac{dv}{dx}$ around:

$$\begin{aligned} \textit{Let} \;\; u &= x & \Longrightarrow \frac{du}{dx} &= 1 \\ \textit{and} \;\; \frac{dv}{dx} &= \sin x & \Longrightarrow v &= -\cos x \end{aligned}$$

Again, plugging this into (1) we get:

$$\int x \sin x \; dx \;=\; -x \cos x \;+\; \int \cos x \; dx$$

Ah, now this is much better because the integral that we are left with on the right is simpler than the one we started with, and in fact is directly integrable to give:

$$\int x \sin x \; dx \;=\; -x \cos x \;+\; \sin x \;+\; C$$

By the way, there are techniques that have been suggested that help you to choose which function should be the u and which the $\frac{dv}{dx}$. One such technique, *LIATE*, is introduced in [Kasube(1983)].

2.3 A Second Example

Let's now integrate

$$I \;=\; \int x^2 \sin x \; dx$$

From what we've already done, we might have an insight that we should choose $u = x^2$ and $\frac{dv}{dx} = \sin x$. So let's do that:

$$\begin{aligned} \textit{Let} \;\; u &= x^2 & \Longrightarrow \frac{du}{dx} &= 2x \\ \textit{and} \;\; \frac{dv}{dx} &= \sin x & \Longrightarrow v &= -\cos x \end{aligned}$$

Plugging this into (1) we get:

$$\int x^2 \sin x \; dx \;=\; -x^2 \cos x \;+\; \int 2x \cos x \; dx \tag{2}$$

The problem we've got now is that the integral on the right-hand side needs to be evaluated. And how are we going to do it? Using IBP, of course! So, to integrate:

$$J \;=\; \int 2x \cos x \; dx$$

we should choose $u = 2x$ and $\frac{dv}{dx} = \cos x$:

$$\begin{aligned} \textit{Let} \;\; u &= 2x & \Longrightarrow \frac{du}{dx} &= 2 \\ \textit{and} \;\; \frac{dv}{dx} &= \cos x & \Longrightarrow v &= \sin x \end{aligned}$$

Plugging this into (1) we get:

$$\int 2x \cos x \; dx \;=\; 2x \sin x \;-\; \int 2 \sin x \; dx$$

and so:

$$\int 2x \cos x \; dx \;=\; 2x \sin x \;+\; 2 \cos x \tag{3}$$

And putting together the results from (2) and (3) we (eventually!) get:

$$\int x^2 \sin x \; dx \;\; = \;\; -x^2 \cos x \; + \; 2x \sin x \; + \; 2 \cos x \; + C$$

or:

$$\int x^2 \sin x \; dx \;\; = \;\; (2 - x^2) \cos x \; + \; 2x \sin x \; + C$$

Phew!

3 An Introduction to DIS: Polynomial and Trigonometrical Functions

So the traditional IBP method is all very well, so why am I interested in another method for doing the same thing? Well actually I'm not. I'm just advocating a different way of looking at IBP: a visual way. A way that emphasises pattern, form and structure. A way that can make the mechanics of IBP easier to control. And a way that can offer an insight into many other areas of mathematics than just integration.

So, by way of an introduction to the tabular, "DIS", method, let's integrate the same

$$I = \int x^2 \;\; \sin x \; dx$$

that we've just integrated the traditional way.

Interestingly, this integral is solved using the tabular method in the film *Stand and Deliver* ([Menéndez(1988)]) by mathematics teacher Jaime Escalante (played by Edward Olmos) of James A. Garfield High School, Los Angeles.

Using DIS, we still have to decide on the u and the dv, and in this case the x^2 factor is the one that we want to differentiate, because it will get simpler. Integrating $\sin x$ will not make it any more complex. Now see Figure 1 for how to lay out the problem. We have a table, with three columns. The D column

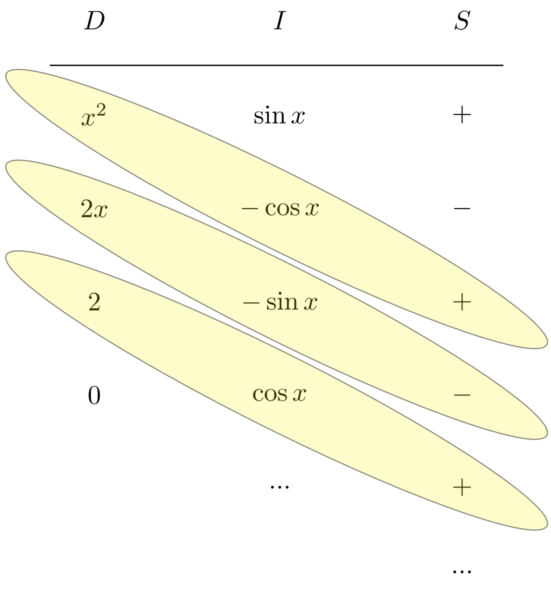

Figure 1: Integrating $\int x^2 \;\; \sin x \; dx$

represents *Differentiation*; the I column represents *Integration* and the S column represents *Sign*. The x^2 goes at the top of the D column, and because it's in the D column we keep differentiating it, row after row, in this case until we get 0. The $\sin x$ goes at the top of the I column, and because it's in the I column we keep integrating it, row after row, in this case until we get down to the line opposite 0 in the D column. The signs in the S column just keep alternating, starting with a +.

Having done all this, we just read down the *diagonals* and multiply the D entry by the I entry by the S entry to give:

$$\begin{aligned} I &= \int x^2 \ \sin x \ dx \\ &= -x^2 \cos x \ + \ 2x \sin x \ + \ 2 \cos x \ + \ C \\ &= (2 - x^2) \cos x \ + \ 2x \sin x \ + \ C \end{aligned}$$

Not forgetting the $+C$ of course. And this is exactly the same result (thankfully!) that we had at the end of section 2.3

And that's it. Now it seems to me as though the DIS layout has two advantages over the traditional layout. Firstly, it's much more concise and simple to manage. And secondly, it emphasises the "series" nature of the IBP (traditional and DIS) method. More on that later!

4 Exponential and Trigonometrical Functions

In section 3 we saw how to use the DIS method to integrate a polynomial multiplied by a periodic function. The sequence terminates after the polynomial differentiates to zero. But what if neither factor differentiates to zero? What if we have something like

$$I = \int e^x \ \sin x \ dx$$

to integrate? This time it turns out that it doesn't really matter which factor we have for D and which for I. See Figure 2 for how to lay out this problem. This time we have to terminate the process somehow,

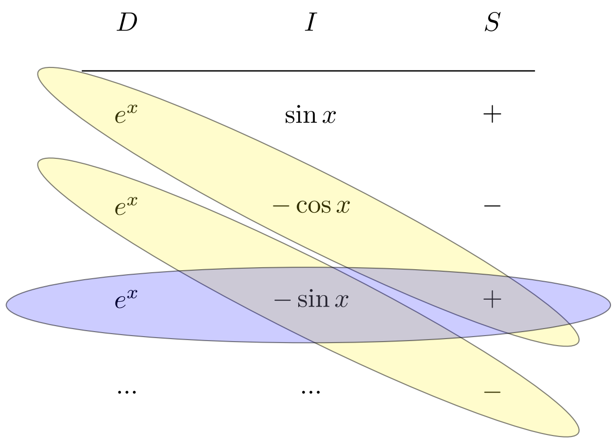

Figure 2: Integrating $\int e^x \ \sin x \ dx$

otherwise it would go on forever. And it turns out that we can terminate the process anytime we like by taking a horizontal slice through our table, and integrating it:

$$\begin{aligned} I &= \int e^x \ \sin x \ dx \\ &= -e^x \cos x \ + \ e^x \sin x \ - \ \int e^x \sin x \ dx \end{aligned}$$

And you might think well, what good is that? We've ended up with the same integral on the right hand side that we started with on the left hand side, so we're no better off. But actually we are better off. Since

$$\int e^x \ \sin x \ dx = -e^x \cos x \ + \ e^x \sin x \ - \ \int e^x \sin x \ dx$$

then we could add $\int e^x \sin x \ dx$ to both sides to give

$$2 \int e^x \ \sin x \ dx = -e^x \cos x \ + \ e^x \sin x$$

and so

$$\int e^x \ \sin x \ dx = \frac{1}{2}\left[-e^x \cos x \ + \ e^x \sin x\right] \ + \ C$$

Not forgetting the $+C$ of course.

And that's it.

Now those of you who are on the ball here might be thinking that we didn't *have* to terminate this process. What if we just carried on...? More on that later!

4.1 Why It Works

To get an idea of why this idea of integrating a horizontal slice through our table can terminate the infinite process, have a look at Figure 3. Here we are using the DIS method on the general case, showing where the IBP formula comes from.

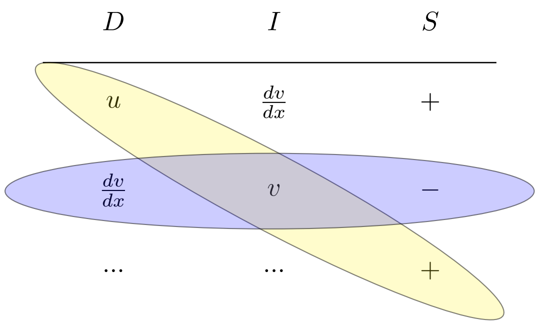

Figure 3: Showing the origin of the IBP formula

Using the DIS method on this example we can see that

$$\int u\frac{dv}{dx} \ dx \ = \ uv \ - \ \int v\frac{du}{dx} \ dx$$

which is just the IBP formula (1) that we know and love so well.

5 Two Trigonometrical Functions

The same idea can be used if we have something like

$$I = \int \sin x \ \cos 2x \ dx$$

to integrate. Again it doesn't really matter which factor we have for D and which for I as they are both periodic functions. See Figure 4 for how to lay out this problem. Again we have to terminate the process after a number of steps by taking a horizontal slice through our table, and integrating it:

$$\begin{aligned} I &= \int \sin x \ \cos 2x \ dx \\ &= \frac{1}{2}\sin x \ \sin 2x \ + \ \frac{1}{4}\cos x \ \cos 2x \ + \ \frac{1}{4}\int \sin x \ \cos 2x \ dx \end{aligned}$$

Then subtracting $\frac{1}{4}\int \sin x \cos 2x \ dx$ from both sides we get

$$\frac{3}{4}\int \sin x \ \cos 2x \ dx = \frac{1}{2}\sin x \ \sin 2x \ + \ \frac{1}{4}\cos x \ \cos 2x$$

and so

$$\begin{aligned} \int \sin x \ \cos 2x \ dx &= \frac{4}{3}\left[\frac{1}{2}\sin x \ \sin 2x \ + \ \frac{1}{4}\cos x \ \cos 2x\right] \ + \ C \\ &= \frac{2}{3}\sin x \ \sin 2x \ + \ \frac{1}{3}\cos x \ \cos 2x \ + \ C \qquad (4) \end{aligned}$$

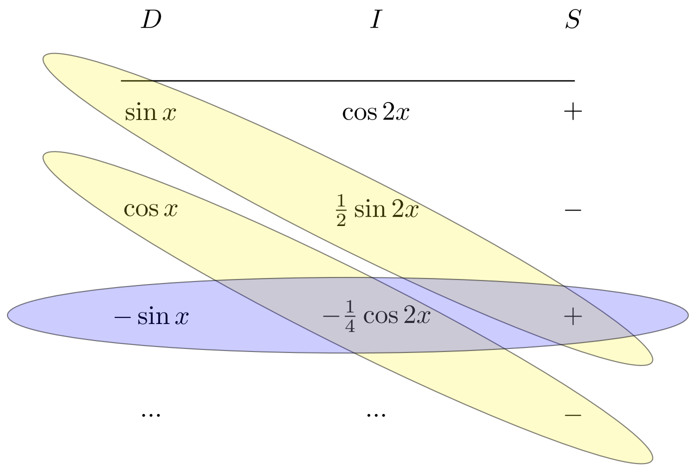

Figure 4: Integrating $\int \sin x \ \cos 2x \ dx$

Not forgetting the $+C$ of course. This is a particularly interesting example because the traditional method of solving this kind of integral is to use trigonometrical double / half angle formulae that are rarely used for anything else. Using the DIS method of IBP, you don't have to be so familiar with those identities. Also, because of the nature of trigonometrical identities, the solution obtained using the traditional method would yield

$$\int \sin x \ \cos 2x \ dx = -\frac{1}{6}\cos 3x \ + \ \frac{1}{2}\cos x \ + \ C \tag{5}$$

and if this was a question in an exam, it would take an eagle-eyed examiner to spot that the student-supplied solution (4) was the same as the solution (5) in her mark scheme.

6 Powers of Sines and Cosines

6.1 The Big Idea

Let's leap straight in here with the idea of trying to integrate a general power of $\sin x$:

$$I_n = \int \sin^n x \ \ dx$$

First of all, take a look at that notation. Here, I_n represents the n^{th} power of $\sin x$. Using the same idea, I_{n-2} would represent the $(n-2)^{th}$ power of $\sin x$, etc, etc.

In order to do this integral, I lay out my table as shown in Figure 5. Again we have to terminate the

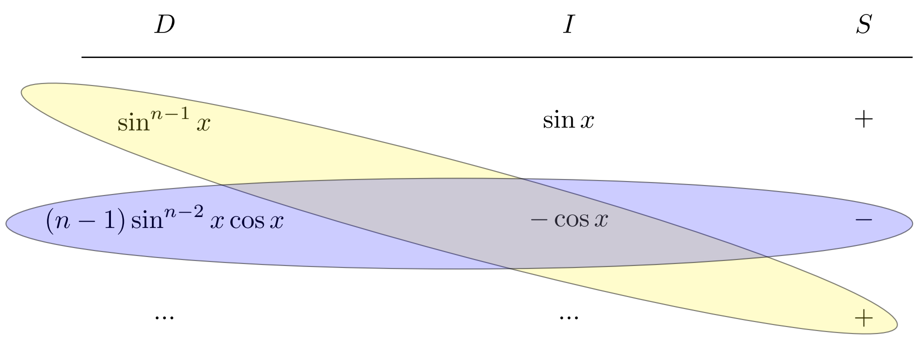

Figure 5: Integrating $\int \sin^n x \ \ dx$

process after a number of steps (this time, only one step!) by taking a horizontal slice through our table, and integrating it:

$$\begin{aligned} I_n &= \int \sin^n x \; dx \\ &= -\sin^{n-1} x \; \cos x \; + \int (n-1)\sin^{n-2} x \; \cos^2 x \; dx \\ &= -\sin^{n-1} x \; \cos x \; + \int (n-1)\sin^{n-2} x \; (1-\sin^2 x) \; dx \end{aligned}$$

because $\sin^2 x + \cos^2 x = 1$. So,

$$\begin{aligned} I_n &= -\sin^{n-1} x \; \cos x \; + \int (n-1)\sin^{n-2} x \; (1-\sin^2 x) \; dx \\ &= -\sin^{n-1} x \; \cos x \; + \int (n-1)\sin^{n-2} x \; dx \; - \int (n-1)\sin^{n-2} x \sin^2 x \; dx \\ &= -\sin^{n-1} x \; \cos x \; + \; (n-1)I_{n-2} \; - \; (n-1)I_n \end{aligned}$$

using that notation idea, and so by adding $(n-1)I_n$ to both sides we get

$$nI_n = -\sin^{n-1} x \; \cos x \; + \; (n-1)I_{n-2}$$

which, by dividing both sides by n, leads to a recurrence relation:

$$I_n \; = \; -\left(\frac{1}{n}\right)\sin^{n-1} x \; \cos x \; + \; \left(\frac{n-1}{n}\right) I_{n-2} \tag{6}$$

that you can use to find the integral of any power of $\sin x$. The same sort of thing can be done for powers of cosines, too.

6.2 A Simple Example

To see how we could use this recurrence relation, lets try integrating

$$I_3 = \int \sin^3 x \quad dx$$

OK, so using our recurrence relation (6), we would get

$$I_3 \; = \; -\frac{1}{3}\sin^2 x \; \cos x \; + \; \frac{2}{3}I_1$$

But I_1 is just the integral of $\sin x$, so this becomes

$$\begin{aligned} I_3 &= -\frac{1}{3}\sin^2 x \; \cos x \; - \; \frac{2}{3}\cos x \\ &= -\frac{1}{3}\cos x \left(\sin^2 x \; + \; 2\right) \end{aligned} \tag{7}$$

Again, the customary A-Level approach to solving this integral leads to the solution

$$I_3 \; = \; -\frac{1}{12}\left(\cos 3x \; - \; 9\cos x\right) \tag{8}$$

and it's pretty hard to spot that equations (7) and (8) are indeed the same.

So you can use DIS to provide an entry into the ideas of using recurrence relations to solve integrals. This used to be on the A-Level maths syllabus, but seems to have fallen by the wayside. Shame.

6.3 A More Difficult Example

This time, lets try integrating

$$I_4 = \int \sin^4 x \quad dx$$

OK, so using our recurrence relation (6), we would get

$$I_4 \; = \; -\frac{1}{4}\sin^3 x \; \cos x \; + \; \frac{3}{4}I_2$$

But we can use the recurrence relation again to find I_2:

$$I_4 = -\frac{1}{4}\sin^3 x\ \cos x + \frac{3}{4}\left(-\frac{1}{2}\sin x\ \cos x + \frac{1}{2}I_0\right)$$

But I_0 is just the integral of 1 ($I_0 = \int sin^0 x dx = \int 1 dx$), so

$$\begin{aligned} I_4 &= -\frac{1}{4}\sin^3 x\ \cos x + \frac{3}{4}\left(-\frac{1}{2}\sin x\ \cos x + \frac{1}{2}x\right) \\ &= -\frac{1}{4}\sin^3 x\ \cos x - \frac{3}{8}\sin x\ \cos x + \frac{3}{8}x \end{aligned}$$

And actually, that wasn't so bad. I hope that you can see how this sort of thing works.

7 Investigating Series

OK, so we can use IBP to solve integrals. Well, after all, that's what it's for, isn't it? True, but that's not the only thing that you can do with it...

7.1 A Series Expansion for e^{-x}

Now at the end of section 4 we had a tantalising glimpse that the DIS layout could lead to an infinite series. This section will explore this possibility.

Let's look at the following integral:

$$I = \int_0^x e^t\ dt$$

The first thing to notice about this integral is that we have limits for the first time. And the second thing is that there isn't a product to integrate! So how do we cope here? Check out Figure 6 .

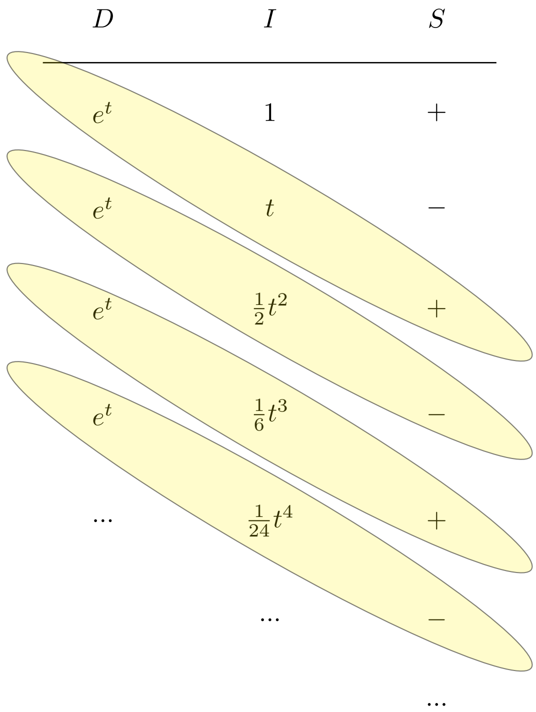

Figure 6: *Integrating* $\int_0^x e^t\ dt$

We have coped with the problem of not having a product by the cunning ploy of making e^t be $e^t \times 1$. And we've put the 1 in the I column.

So since

$$\begin{aligned} I &= \int_0^x e^t \; dt \\ &= e^x \; - \; 1 \end{aligned}$$

by ordinary integration, and by using DIS,

$$\begin{aligned} I &= \int_0^x e^t \; dt \\ &= \Big[te^t \; - \; \frac{1}{2}t^2e^t \; + \; \frac{1}{6}t^3e^t \; - \; \frac{1}{24}t^4e^t ... \Big]_0^x \\ &= \Big[e^x \Big(\; x \; - \; \frac{1}{2}x^2 \; + \; \frac{1}{6}x^3 \; - \; \frac{1}{24}x^4 ... \Big) \Big] - 0 \end{aligned}$$

then we have obtained

$$e^x \; - \; 1 \; = \; e^x \Big(\; x \; - \; \frac{1}{2}x^2 \; + \; \frac{1}{6}x^3 \; - \; \frac{1}{24}x^4 \; ... \Big)$$

Now by dividing both sides by e^x and rearranging, we get

$$e^{-x} = \; 1 \; - \; x \; + \frac{1}{2}x^2 \; - \frac{1}{6}x^3 + \frac{1}{24}x^4 \; ...$$

So the DIS layout of IBP has led us directly to a series expansion of e^{-x}! That's pretty cool. And by similar means it's possible to get series expansions of e^x ([Chamberland(2008)]), *sinx* and *cosx* ([Johnson(1984)]) and all kinds of other stuff using DIS (see section 9). Normally you need Taylor's Theorem to do this kind of thing...

7.2 A Derivation of Taylor's Theorem(!)

Consider the integral

$$I = \int_a^x f'(t) \; dt$$

Well, by defintion[2] this is just

$$f(x) \; - f(a)$$

Interestingly, we can use the DIS method to attack this. And there are a few quirks of the technique that are quite illuminating. Firstly, we can use that old cunning ploy of the multiplying by 1 thing to give us a product: this time I'm going to have the product of multiplying $-f'(t)$ and -1 together. You don't have to do it this way, with all these minus signs, but I think it makes the algebra slightly easier to follow. Secondly, if you look at the DIS layout for this integral (see Figure 7) then when you integrate the -1, in going from the first to the second line of the layout, any integral of -1 will do. Usually, that would mean $-t$, but of course, $-t + C$ is OK too (C just being any constant). And for reasons that (it is to be hoped!) will become clearer later, my choice of constant is x.

OK, so if you follow through with the DIS method, you should now end up with:

$$f(x) \; - \; f(a) \; = \; \Big[-(x-t)f'(t) \; - \; \frac{1}{2}(x-t)^2 f''(t) \; - \; \frac{1}{6}(x-t)^3 f'''(t) \; - \; \frac{1}{24}(x-t)^4 f^{iv}(t) \; ... \Big]_a^x$$

Now here's the cunning bit. The reason that I chose my constant of the inetgration of -1 in the first line of the layout to be x is so that now, when we insert the limits into the above integral, all the terms from the upper limit vanish:

$$\begin{aligned} f(x) \; - \; f(a) \; = & \Big[-(x-x)f'(x) \; - \; \frac{1}{2}(x-x)^2 f''(x) \; - \; \frac{1}{6}(x-x)^3 f'''(x) \; - \; \frac{1}{24}(x-x)^4 f^{iv}(x) \; ... \Big] \\ & - \Big[-(x-a)f'(a) \; - \; \frac{1}{2}(x-a)^2 f''(a) \; - \; \frac{1}{6}(x-a)^3 f'''(a) \; - \; \frac{1}{24}(x-a)^4 f^{iv}(a) \; ... \Big] \end{aligned}$$

[2] Actually, from the statement of the fundamental theorem of calculus!

D	I	S
$-f'(t)$	-1	$+$
$-f''(t)$	$x-t$	$-$
$-f'''(t)$	$-\frac{1}{2}(x-t)^2$	$+$
$-f^{iv}(t)$	$\frac{1}{6}(x-t)^3$	$-$
...	$-\frac{1}{24}(x-t)^4$	$+$
	...	$-$
		...

Figure 7: A derivation of Taylor's Theorem!

and so finally

$$f(x) \;=\; f(a) \;+\; (x-a)f'(a) \;+\; \frac{1}{2}(x-a)^2 f''(a) \;+\; \frac{1}{6}(x-a)^3 f'''(a) \;+\; \frac{1}{24}(x-a)^4 f^{iv}(a) \; ...$$

So you can use DIS to prove Taylor's Theorem. And consequently MacLaurin's Theorem, by putting 0 for a.

See also [Lampret(2001)] for more of this sort of thing.

8 Enrichment Activities

Now it strikes me that if you were looking for an area of mathematics that provided opportunities for independent research for 16-18 year-olds, I would have thought that the DIS view of IBP was right up your street. It's quite exciting to come up with something original, something you haven't learnt from a book. And with DIS anyone doing A-Level can do this.

As an example, let's have a look at the integral

$$I = \int \frac{x^3}{(1+x)^5}\, dx$$

The usual way of solving this integral would be to use the substitution $u = 1 + x$. Doing this, or using the Wolfram Mathematica Online Integrator ([Wolfram Research(1996+)]) you would end up with the solution

$$I = -\frac{(2x+1)(2x(1+x)+1)}{4(1+x)^4}$$

However, using DIS (see Figure 8) you would obtain

$$I = -\frac{1}{4}\frac{x^3}{(1+x)^4} - \frac{3}{12}\frac{x^2}{(1+x)^3} - \frac{6}{24}\frac{x}{(1+x)^2} - \frac{6}{24}\frac{1}{(1+x)}$$

D	I	S
x^3	$(1+x)^{-5}$	$+$
$3x^2$	$-\frac{1}{4}(1+x)^{-4}$	$-$
$6x$	$\frac{1}{12}(1+x)^{-3}$	$+$
6	$-\frac{1}{24}(1+x)^{-2}$	$-$
0	$\frac{1}{24}(1+x)^{-1}$	$+$
	...	$-$
		...

Figure 8: *Integrating* $\int \frac{x^3}{(1+x)^5}\ dx$

or

$$I = -\frac{1}{4}\frac{1}{(1+x)}\left[1+\left(\frac{x}{1+x}\right)+\left(\frac{x}{1+x}\right)^2+\left(\frac{1}{1+x}\right)^3\right]$$

where the structure of the solution is much more apparent.

You might then have the idea of generalising the integral. Perhaps to

$$I = \int \frac{x^n}{(1+x)^{n+2}}\ dx$$

and if you do this (see Figure 9) you end up with

$$\begin{aligned} I = &-\frac{1}{(n+1)}\frac{x^n}{(1+x)^{n+1}} - \frac{n}{n(n+1)}\frac{x^{n-1}}{(1+x)^n} - \frac{n(n-1)}{(n-1)n(n+1)}\frac{x^{n-2}}{(1+x)^{n-1}} \\ &- \frac{n(n-1)(n-2)}{(n-2)(n-1)n(n+1)}\frac{x^{n-3}}{(1+x)^{n-2}} - \ \dots \end{aligned}$$

or, realising that there will be a finite number of terms (n+1, actually) since eventually x^n differentiates to zero, we get

$$I = -\frac{1}{(n+1)}\frac{1}{(1+x)}\left[1+\left(\frac{x}{1+x}\right)+\left(\frac{x}{1+x}\right)^2+\left(\frac{x}{1+x}\right)^3+\ \dots\ +\left(\frac{x}{1+x}\right)^n\right]$$

which is actually very pretty. Not only that, as it contains a geometric series, we can simplify the sum to

$$I = \frac{1}{(n+1)}\left[\ \left(\frac{x}{1+x}\right)^{n+1}\ -\ 1\ \right]$$

which is really rather neat.

D	I	S
x^n	$(1+x)^{-(n+2)}$	$+$
nx^{n-1}	$-\frac{1}{n+1}(1+x)^{-(n+1)}$	$-$
$n(n-1)x^{n-2}$	$\frac{1}{n(n+1)}(1+x)^{-n}$	$+$
$n(n-1)(n-2)x^{n-3}$	$-\frac{1}{(n-1)n(n+1)}(1+x)^{-(n-1)}$	$-$
...	$\frac{1}{(n-2)(n-1)n(n+1)}(1+x)^{-(n-2)}$	$+$
	...	$-$
		$+$

Figure 9: *Integrating* $\int \frac{x^n}{(1+x)^{n+2}}\ dx$

The point of all this is that even by picking an integral almost at random, it was possible to show some delightful structure in the solution of that integral. And to reiterate, isn't mathematics all about beauty, pattern, form and structure? And the DIS layout leads directly to structures, forms, patterns and beauty that might not be apparent using other methods.

9 Literature Search

9.1 General

The earliest use of the tabular method I've found so far is [Folley(1947)]. [Brown(1960)] develops the tabular method in an algebraic form through induction. [Murty(1980)] provides a good introduction to the tabular method as early as 1980. [Kasube(1983)] discusses the LIATE acronym for deciding the allocation of products to u and dv. [Gillman(1991)] follows up on the paper [Horowitz(1990)], which is the most cited article in this field, to my knowledge. He shows how to produce infinite series, Laplace transforms, Taylor's formula, the residue theorem for meromorphic functions. An important paper.

The only books that I have come across that contain examples of the tabular method are [Foerster(2005)] and [Zegarelli(2008)]. [Foerster(2005)] is also the only reference I have found to the idea of transference. [Sheard(2009)] develops the ideas of the "n-step algebra trick".

9.2 IBP and Linear Algebra

[Rogers Jr(1997)] uses IBP to help solve calculus problems using linear algebra.

9.3 IBP and Infinite Series

[Johnson(1984)] develops derivations of power series in an alternative way to Taylor's theorem. [Chamberland(2008)] develops an infinite series for e using IBP. [Kilmer(2008)] shows how to obtain infinite series using IBP.

9.4 Solving Differential Equations using IBP

[Reut(1995)] solves differential equations using repeated IBP and series.

9.5 Products of Sines and Cosines

[Nicol(1993)] integrates products of sines and cosines using IBP, along similar lines to those adopted in this document.

9.6 Fourier Series and Laplace Transforms

As Fourier Series and Laplace Transforms are integrations of products, the DIS approach is an ideal tool to attack problems of this type. [Khattri(2008)] provides examples.

9.7 IBP and Elliptic Integrals(!)

[Olds(1949)] links IBP to elliptic integrals, and more. He does this by first introducing the idea that from the IBP formula

$$\int u\frac{dv}{dx}\,dx \;=\; uv \;-\; \int v\frac{du}{dx}\,dx$$

by treating the differentials as fractions, this could simplify to

$$\int u\;dv \;=\; uv \;-\; \int v\;du \tag{9}$$

which is another interesting twist on IBP! This means that if you want to find the integral of a function, you can get it by integrating its inverse function! And depending on circumstances, that could be easier. For example, if we wanted to integrate

$$I \;=\; \int \arcsin x\;dx$$

then from (9) if we let $y = \arcsin x$, so that $x = \sin y$ we can write this as

$$\begin{aligned} I &= \int \arcsin x\;dx \;=\; \int y\;dx \\ &= xy \;-\; \int \sin y\;dy \\ &= x\arcsin x \;+\; \cos y \end{aligned}$$

and then all that remains is to show that $\cos y = \cos(\arcsin x) = \sqrt{1-x^2}$. This can be done by realising that if $x = \sin y$ then $\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$. Job done. And that's a lot easier than the standard method currently used in Further Maths Units, which uses IBP(!) on the product $\arcsin x \times 1$ when you then have to differentiate the $\arcsin x$...

Another fine example of the use of (9) is integrating $\ln x$. If we let $y = \ln x$, then $x = e^y$, and:

$$\begin{aligned} I &= \int \ln x \; dx \;=\; \int y \; dx \\ &= xy \;-\; \int x \; dy \\ &= xy \;-\; \int e^y \; dy \\ &= x \ln x \;-\; e^y \\ &= x \ln x \;-\; x \\ &= x(\ln x \;-\; 1) \end{aligned}$$

Well that was pretty simple. And no tricks!

9.8 A Bit of Fun

[Walsh(1927)] came up with a neat little paradox resulting from IBP. Let's say we wanted to evaluate $\int \frac{1}{x} dx$. We could set out the problem as shown in Figure 10.

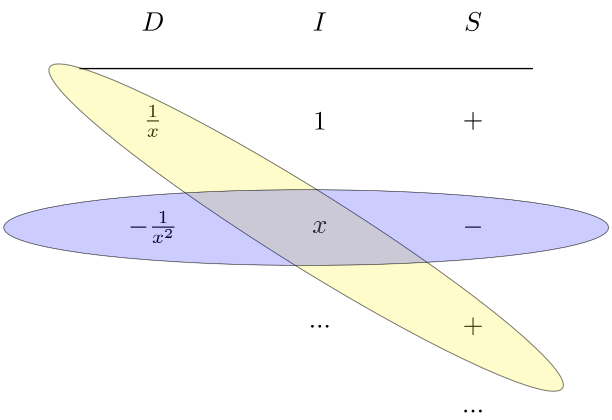

Figure 10: Integrating $\int \frac{1}{x} dx$

which leads to the intriguing equation:

$$\int \frac{1}{x} \; dx \;=\; 1 \;+\; \int \frac{1}{x} \; dx$$

So - what's going on here, then?

10 Conclusion

If you only use DIS to evaluate integrals, fine. The DIS method IS the traditional method of IBP, but cut down to its essentials. It is simply presented in a form that makes the process as easy to follow as possible, in an accessible, visual way. For that use alone, I would recommend it.

But you can use DIS to do so much more. It really is a wonderful tool that has completely unexpected uses.

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