Steve Smith Tuition: Maths Notes

Integration by Parts, the Tabular Method, I: "DIS is how you do it!"

Contents

Prerequisites

It is assumed that the reader is aware of the product rule for differentiation. It would be nice if the reader is also familiar with Integration By Parts as a tool to integrate the product of two functions, but I explain the idea of the technique in Section 2, so this document is self-contained.

Notes

There is a companion document to this one, *Integration by Parts, the Tabular Method, II: or "DIS is how you do more with it!"*, which is aimed at undergraduate standard students, and teachers. That outlines the DIS approach to Integration By Parts and provides an insight into many areas other than integration where Integration By Parts can be used.

Document History

Date	Version	Comments
3rd April 2011	1.0	Initial creation of the document.
21st February 2012	2.0	Updating to include lots of examples.
30th July 2012	3.0	Showing how DIS can produce the IBP formula.
21st January 2016	4.0	Overhauling the document structure, and changing the font to Palatino.

1 Introduction

1.1 Integration as a Whole

You can only integrate what I call *standard forms*. These are functions that you just know the integrals of. They are primarily things like: polynomial terms ax^n; trigonometrical functions like $\sin(x)$ and $\cos(x)$; and other assorted things like e^x and $\ln(x)$. See Appendix A for a more extensive list of standard forms.

Quite often, you are given a formula sheet to take into an exam on integration, and in the formula sheet will be a list of functions and their integrals. In that case, everything in the list will be a standard form, as if you needed to integrate a function in the list, you can just write down the answer. There's no working to do.

So, what happens when you get an integral that's *not* a standard form?

Well, there's only *four* things you can do[1]. Each one of these four techniques is used to transform your integral into a standard form. The four techniques are:

- partial fractions;
- use of trigonometrical identities;
- substitution;
- integration by parts.

That's all.

Only those four. Nothing else.

In this document we will be looking at integration by parts. I discuss integration by substitution in Smith (2016).

1.2 Integration by Parts

Students often find Integration By Parts (IBP) tricky. It's not that the ideas are hard, it's more that the details of the method are finicky to control, and it's easy to make a slip with a minus sign, or by integrating something that you should be differentiating, etc.

Wouldn't it be nice if there was an alternative, simple, *visual* way of doing IBP that minimised the opportunities for mistakes? Well, there is.

In this document I have called it the "DIS" method. I like that name, because when a student has trouble with IBP, I would say: "OK: DIS is how you do it!". The more usual name is the *Tabular Integration By Parts* method, and the literature follows that name. In Indonesia it seems to be called the *Tanzalin* method.

This method does not seem to be widely known, and is *not* taught in schools. It should be. Spread the word!

But before we get to the fun bit, let's have a quick review of the traditional IBP method...

[1] I'm talking here about *algaebraic* integration. You could always use *numerical* integration to integrate *any* function, even standard forms. That's a subject for another day...

2 The Traditional Integration by Parts

2.1 Deriving the Formula

We start with the product rule for differentiation. If u and v are both functions of x, then the product rule states:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Now because this is an equation, and you can do anything you like to equations[2], then we could integrate both sides of this equation with respect to x:

$$\int \frac{d}{dx}(uv)\ dx = \int u\frac{dv}{dx}\ dx + \int v\frac{du}{dx}\ dx$$

But the left-hand side of this equation is the integral of the differential of something. And integration and differentiation are inverse processes, so the left-hand side simplifies to just uv:

$$uv = \int u\frac{dv}{dx}\ dx + \int v\frac{du}{dx}\ dx$$

And subtracting $\int v\frac{du}{dx}\ dx$ from both sides gives the integration by parts formula that we all know and love:

$$\int u\,\frac{dv}{dx}\,dx = uv - \int v\,\frac{du}{dx}\,dx \tag{1}$$

2.2 First Example

By way of an introductory example, let's integrate

$$I = \int x\sin(x)\ dx$$

Now from (1), we need to choose which factor of our product will be the u and which will be the $\frac{dv}{dx}$. This is a very important step in this method, because your choice is crucial.

Here's the thinking. If you look at the right-hand side of (1) you see that u appears, as does $\frac{du}{dx}$. So to use the formula, we need to differentiate u. Also by inspection of the right-hand side of (1), v appears. That means we are going to have to integrate $\frac{dv}{dx}$. So we are going to have to differentiate one of the functions of our product and integrate the other. Also, on the right-hand side of (1) we have another integral to do. And for this entire process to be useful, *that integral needs to be simpler than the one we started with*!

OK, so with all that in mind, let's choose $u = \sin(x)$ and $\frac{dv}{dx} = x$ and let's see what happens:

$$\text{Let } u = \sin(x) \qquad \Rightarrow \frac{du}{dx} = \cos(x)$$
$$\text{and } \frac{dv}{dx} = x \qquad \Rightarrow v = \tfrac{1}{2}x^2$$

[2] So long as you do the same thing to both sides!

So, plugging all this in to (1) we get:

$$\int x\sin(x)\ dx = \tfrac{1}{2}x^2\sin(x) - \int \tfrac{1}{2}x^2\cos(x)\ dx$$

which is all very well, but the problem is that the integral on the right-hand side ($\int \frac{1}{2}x^2\cos(x)dx$) is now *worse* than the one we started with ($\int x\sin(x)dx$), because of the x^2 bit.

Now let's have a look at what happens when we swap the u and $\frac{dv}{dx}$ around:

$$\begin{aligned} \text{Let } u &= x & \Rightarrow \frac{du}{dx} &= 1 \\ \text{and } \frac{dv}{dx} &= \sin(x) & \Rightarrow v &= -\cos(x) \end{aligned}$$

Again, plugging this into (1) we get:

$$\int x\sin(x)\ dx = -x\cos(x) + \int \cos(x)\ dx$$

Ah, now this is much better because the integral that we are left with on the right is simpler than the one we started with, and in fact is directly integrable to give:

$$\int x\sin(x)\ dx = -x\cos(x) + \sin(x) + C$$

By the way, there are procedures that have been suggested that help you to choose which function should be the u and which the $\frac{dv}{dx}$. One such scheme, *LIATE*, is introduced in Kasube (1983). Hopefully, as you work through this document you will get an idea of how to do the choosing.

2.3 A Second Example

Let's now integrate

$$I = \int x^2\sin(x)\ dx$$

From what we've already done, we might have an insight that we should choose $u = x^2$ and $\frac{dv}{dx} = \sin(x)$. So let's do that:

$$\begin{aligned} \text{Let } u &= x^2 & \Rightarrow \frac{du}{dx} &= 2x \\ \text{and } \frac{dv}{dx} &= \sin(x) & \Rightarrow v &= -\cos(x) \end{aligned}$$

Plugging this into (1) we get:

$$\int x^2\sin(x)\ dx = -x^2\cos(x) + \int 2x\cos(x)\ dx \tag{2}$$

The problem we've got now is that the integral on the right-hand side needs to be evaluated. And how are we going to do it? Using IBP, of course! So, to integrate:

$$J = \int 2x\cos(x)\ dx$$

we should choose $u = 2x$ and $\frac{dv}{dx} = \cos(x)$:

$$\begin{aligned} \text{Let } u &= 2x & \Rightarrow \frac{du}{dx} &= 2 \\ \text{and } \frac{dv}{dx} &= \cos(x) & \Rightarrow v &= \sin(x) \end{aligned}$$

Plugging this into (1) we get:

$$\int 2x\cos(x)\ dx = 2x\sin(x) - \int 2\sin(x)\ dx$$

and so:

$$\int 2x\cos(x)\ dx = 2x\sin(x) + 2\cos(x) \tag{3}$$

And putting together the results from (2) and (3) we (eventually!) get:

$$\int x^2\sin(x)\ dx = -x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C$$

or:

$$\int x^2\sin(x)\ dx = (2 - x^2)\cos(x) + 2x\sin(x) + C$$

Phew!

3 But...*DIS* is How You Do It!

3.1 A Quick Outline...

The traditional IBP method is all very well, and works fine. So why am I interested in *another* method for doing the same thing?

Well actually, I'm not. I'm just advocating a different way of looking at IBP: a *visual* way. A way that emphasises *pattern*, *form* and *structure*. A way that makes the mechanics of IBP easier to control. A way that reduces the number of mistakes you make. And a way that can offer an insight into many other areas of mathematics than just integration. Sound good?

So, by way of an introduction to the tabular, "DIS", method, let's integrate the same

$$I = \int x^2 \sin(x)\ dx$$

that we've just integrated the traditional way[3].

Using DIS, we still have to decide on the u and the dv, and in this case the x^2 factor is the one that we want to differentiate, because it will get simpler. Integrating $\sin(x)$ will not make it any more complex.

Now have a look at Figure 1 for a visual way to lay out the problem.

D	I	S
x^2	$\sin(x)$	$+$
$2x$	$-\cos(x)$	$-$
2	$-\sin(x)$	$+$
0	$\cos(x)$	$-$
		$+$

Figure 1: Integrating $\int x^2 \sin x\ dx$

We have a table with three columns, labelled D, I and S. The D column represents *Differentiation*, the I column represents *Integration*, and the S column represents *Sign* (that is, $+$ or $-$).

We're going to put the x^2 at the top of the D column (because we've decided to differentiate the x^2), and because it's in the D column we keep differentiating it, row after row, in this case until we get 0.

We put the $\sin(x)$ at the top of the I column. And because it's in the I column we keep integrating it, row after row, in this case until we get down to the line opposite 0 in the D column.

The signs in the S column just keep *alternating*, starting with a $+$.

Once you've filled in the table, mark off the *diagonals*, starting with the top-left entry in the table, as shown in Figure 1.

And we're nealy done! To find the integral, we just read down the *diagonals* and multiply the D entry by the I entry by the S entry for each diagonal to give exactly the same result (thankfully!) that we had at the

[3]Interestingly, this integral is solved using the tabular method in the film *Stand and Deliver* (Menèndez (1988)) by mathematics teacher Jaime Escalante (played by Edward Olmos) of James A. Garfield High School, Los Angeles.

end of section 2.3:

$$\begin{aligned} I &= \int x^2 \sin(x)\ dx \\ &= -x^2 \cos(x) + 2x \sin(x) + 2\cos(x) + C \\ &= (2 - x^2)\cos(x) + 2x \sin(x) + C \end{aligned}$$

3.2 ...And in a A Bit More Detail

Let's go through that again, really slowly, so that you get the idea.

We want to try and integrate

$$I = \int x^2 \sin(x)\ dx$$

Picking the Thing to Differentiate

The first thing we have to do is to decide which thing we're going to differentiate (the u in the traditional approach), and which thing we're going to integrate (the $\frac{dv}{dx}$ from the traditional approach). I'm going to talk more about how to decide which you pick for which a bit later (see Section 4). For now, I'll tell you that it's best to differentiate x^2 and integrate $\sin(x)$. So, having made that choice, we can construct our table and fill in the first row. The entry in the first row of the S column is *always* $+$.

D	I	S
x^2	$\sin(x)$	$+$

Figure 2: Integrating $\int x^2 \sin x\ dx$: The First Row

Do the Differentiating and the Integrating

Right, now in the D column, we have to keep differentiating. How do you know when to stop? Stop when you get 0. Why? I'll tell you in a minute.

D	I	S
x^2	$\sin(x)$	$+$
$2x$		
2		
0		

Figure 3: Integrating $\int x^2 \sin x\ dx$: After Differentiating

Next we deal with the I column. Now in the I column, we have to keep integrating. How do you know when to stop? Stop when you get to the row with the 0 in the D column. Why? I'll tell you in a minute.

D	I	S
x^2	$\sin(x)$	$+$
$2x$	$-\cos(x)$	
2	$-\sin(x)$	
0	$\cos(x)$	

Figure 4: Integrating $\int x^2 \sin x \, dx$: After Integrating

And finally we can fill in the S column. The entries in the S column are either $+$ or $-$, and they *alternate*. Why? I'll tell you in a minute. And when do you stop? In the row *after* the D and I columns finish. Why? I'll tell you in a minute.

D	I	S
x^2	$\sin(x)$	$+$
$2x$	$-\cos(x)$	$-$
2	$-\sin(x)$	$+$
0	$\cos(x)$	$-$
		$+$

Figure 5: Integrating $\int x^2 \sin x \, dx$: After Filling in the Sign Column

Write Down the Answer

To find the answer to the integral, what we do now is to look down *diagonals*. See Figure 6 for where to draw the *first* diagonal.

D	I	S
x^2	$\sin(x)$	$+$
$2x$	$-\cos(x)$	$-$
2	$-\sin(x)$	$+$
0	$\cos(x)$	$-$
		$+$

Figure 6: Integrating $\int x^2 \sin x \, dx$: First Diagonal

The first diagonal starts at the top left corner of the table, and goes down to the right, incorporating one entry from each column (Figure 6).

What we do now is to *multiply* the entries in the diagonal together, and the result will be one of the terms of our answer! So, for the first diagonal,

$$\text{first diagonal term} = x^2 \times -\cos(x) \times +$$

so that the term we get from the first diagonal will be $-x^2\cos(x)$.

In a similar way, the second diagonal will yield the term $+2x\sin(x)$ when we multiply the entries together (see Figure 7). Why is it $+$? Because there are *two* negative signs in this diagonal, and when you multiply two negative signs together, you get a positive result.

D	I	S
x^2	$\sin(x)$	$+$
$2x$	$-\cos(x)$	$-$
2	$-\sin(x)$	$+$
0	$\cos(x)$	$-$
		$+$

Figure 7: Integrating $\int x^2 \sin x \; dx$: Second Diagonal

When we do this for *all* the diagonals (see Figure 8)

D	I	S
x^2	$\sin(x)$	$+$
$2x$	$-\cos(x)$	$-$
2	$-\sin(x)$	$+$
0	$\cos(x)$	$-$
		$+$

Figure 8: Integrating $\int x^2 \sin x \; dx$: All Diagonals

we end up with the result

$$I = -x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C$$

not forgetting the $+C$, of course.

So once we've drawn up the table, you can just read the answer from it, by looking down the diagonals.

3.3 Answers? Questions! Questions? Answers!

Let's start answering some of those questions that I've posed in the previous section.

How Do I Choose Which Thing to Differentiate?

This I'm going to cover in depth in Section 4.

How Do I Know When To Stop Differentiating?

When you've picked the thing to differentiate (the x^2 in this example), then how do you know when to stop differentiating it? Well, the key lies in what you do with the diagonals. Remember that to get our answer, we need to *multiply* the things in the diagonals. And of course when you multiply something by zero, you get zero. So if anything in a diagonal was zero, then the whole diagonal will be be zero, and we can ignore it, right?

So as soon as you have got a zero in the D column you can stop, because every diagonal from there on will contribute exactly zero to the final integral.

How Do I Know When To Stop Integrating?

You can stop integrating the thing at the top of the I column when you get to an entry of 0 in the D column since if you integrated it again, that value would be in a diagonal with a 0 in the D column, which we have just seen that we can ignore. And there will be 0 in the D column forever after, so there's no point in integrating anything in the I column after you've got to the a 0 in the D column.

Why Does the Sign Column Start With a $+$, and Why Does the Sign Alternate?

I'm going to leave answering these question until Section 5.4. I hope that's OK.

3.4 Another Example

OK, let's see if we can use DIS to find this integral:

$$I = \int xe^x \, dx$$

This time, I'm going to put x in the D column, and e^x in the I column. The reason for this choice will be explained in Section 4. Here's the table:

D	I	S
x	e^x	$+$
1	e^x	$-$
0	e^x	$+$
		$-$

Figure 9: Integrating $\int xe^x \, dx$

So, multiplying down the diagonals, our integral will be

$$\begin{aligned} I &= xe^x - e^x + C \\ &= e^x(x-1) + C \end{aligned}$$

3.5 The Vast Majority of Cases

Ninety-nine times out of ten, if you get an integration by parts in an A-Level exam, you will get one of the following integrals

$$\begin{aligned} \int x\sin(x)\;dx &\qquad \text{or} \qquad \int x^2\sin(x)\;dx \\ \int x\cos(x)\;dx &\qquad \text{or} \qquad \int x^2\cos(x)\;dx \\ \int xe^x\;dx &\qquad \text{or} \qquad \int x^2e^x\;dx \end{aligned}$$

And if you do, you can use the techniques described above in Sections 3.2 and 3.4. Simples.

4 How to Choose the u and the $\frac{dv}{dx}$

Now if you've been paying attention, you might have come up with a reason why we chose x^2 for the u in Section 3.2, and x for the u in Section 3.4.

It's to do with this business of being able to ignore all the diagonals that have a 0 in them. Since we multiply all the terms in a diagonal to get a term for the integral, then any diagonal with a zero in it can be ignored.

OK, so how can we ensure that we get a zero somewhere in a diagonal? Well, if our product has a polynomial term in it (x, x^2, etc), then if we put that in the D column, that will eventually differentiate away to zero!

So, if you have any of the integrals listed in Section 3.5 (or something like those with a different power of x), then the x^n term will be your u (so that goes in the D column), and the $\sin(x)$, $\cos(x)$ or e^x term will be the $\frac{dv}{dx}$ (so that goes in the I column).

4.1 Possible Problems...

But hang on a minute, I hear you cry. What happens if you don't have something in your product that will differentiate away to nothing? Something like this, for example:

$$I = \int e^x \sin(x)\ dx$$

Now with this integral, when you draw up the table and fill it in by differentiating the D stuff and integrating the I stuff, you realise quite quickly that *you will go on differentiating and integrating forever*. Your integral turns into *an infinite series*. Wow. Wasn't expecting that!

Or maybe you have something like this

$$I = \int x^2 \ln(x)\ dx$$

which looks innocuous, but when we try to use our method, we run into a snag. The thing is that from what we've done so far, we'd want to put the x^2 in our D column, so that it differentiates away to nothing. So far so good. But then the $\ln(x)$ would have to go in the I column. But here's the snag: we don't know how to integrate $\ln(x)$!!

Rats.

Now these problems aren't problems with DIS: they are problems encountered in IBP *however* you do it. So you will run into situations like these whether you use DIS or the traditional method or any other method.

So what can we do? Well, wouldn't it be nice if we didn't have to rely on the things in the D column differentiating away to zero? Wouldn't it be nice if there was a way where it didn't matter what you put in what column: there was *always* a way of terminating the process whenever we like? Wouldn't it be nice if we were in control of that infinite series, and could stop it whenever we wanted?

If only...

5 Terminating the Infinite Process

5.1 The Basic Idea

In Section 3 we saw how to use the DIS method to integrate things like $x^2 \sin(x)$ and $x^2 e^x$. With these kinds of integrals we put the polynomial bit in the D column, and when we differentiate it continually, it eventually turns into zero, and we can stop because we can ignore all the diagonals from there on.

But what if neither factor differentiates to zero? What if we have something like

$$I = \int e^x \sin(x)\, dx$$

to integrate? This time it turns out that it doesn't really matter which factor we have for D and which for I. See Figure 10 for what happens when we lay out this problem.

D	I	S
e^x	$\sin(x)$	$+$
e^x	$-\cos(x)$	$-$
e^x	$-\sin(x)$	$+$
e^x	$\cos(x)$	$-$
...	...	...

Figure 10: Integrating $\int e^x \sin(x)\, dx$

So, what do we do? Well, it turns out that there is a way to terminate this process! How? We can terminate the process any time we like by *taking a horizontal slice through our table, and integrating it*. That's how.

To find our integral, we start by putting in the diagonals, as before,

D	I	S
e^x	$\sin(x)$	$+$
e^x	$-\cos(x)$	$-$
e^x	$-\sin(x)$	$+$
		$-$

Figure 11: Integrating $\int e^x \sin(x)\, dx$: Starting the Process

but instead of doing this forever, we terminate the process by taking a horizontal slice, starting with an entry in the D column *that we haven't used yet*. What I mean by that is that the first two e^x terms in Figure 11 have been used in the first two diagonals. But the third one hasn't, so we can take a horizontal slice on the third row (see Figure 12).

Now that taking-a-horizontal-slice thing terminates the process. Why? I'll tell you in a minute. But for now, what we need to realise is that our integral can be expressed like this

$$\begin{aligned} I &= \int e^x \sin(x)\, dx \\ &= -e^x \cos(x) + e^x \sin(x) - \int e^x \sin(x)\, dx \end{aligned}$$

D	I	S
e^x	$\sin(x)$	$+$
e^x	$-\cos(x)$	$-$
e^x	$-\sin(x)$	$+$
		$-$

Figure 12: Integrating $\int e^x \sin(x)\ dx$: Taking a Horizontal Slice!

where the first two terms are the multiplying-down-the-diagonal things that we had before, and the third term is our integrating-the-horizontal-slice thing.

And you might think well, what good is that? We've ended up with the same integral on the right hand side that we started with on the left hand side, so we're no better off. But actually we *are* better off. Since

$$\int e^x \sin(x)\ dx = -e^x \cos(x) + e^x \sin(x) - \int e^x \sin(x)\ dx$$

then we could add $\int e^x \sin x\ dx$ to both sides to give

$$2\int e^x \sin(x)\ dx = -e^x \cos(x) + e^x \sin(x)$$

and so

$$\int e^x \sin(x)\ dx = \tfrac{1}{2}\Big[-e^x \cos(x) + e^x \sin(x)\Big] + C$$

Not forgetting the $+C$ of course.

5.2 The Other Kind of Problem...

In Section 4.1 I mentioned another kind of integral that we might have a problem with. Consider

$$\int x^2 \ln(x)\ dx$$

If we start up a table for this integral, we have a choice of what to put in what columns. Let's say we tried putting x^2 in the D column and $\ln(x)$ in the I column. We then get

D	I	S
x^2	$\ln(x)$	$+$
$2x$	???	$-$

Figure 13: Integrating $\int x^2 \ln(x)\ dx$: First Attempt...

where we run into the immediate problem of integrating $\ln(x)$.

Or we could put $\ln(x)$ in the D column and x^2 in the I column. We then get

D	I	S
$\ln(x)$	x^2	$+$
$\frac{1}{x}$	$\frac{1}{3}x^3$	$-$
$-\frac{1}{x^2}$	$\frac{1}{12}x^4$	$+$
...	...	...

Figure 14: Integrating $\int x^2 \ln(x)\ dx$: Second Attempt...

which runs into that problem of the infinite series, as both D and I columns would go on forever.

Ah! But we could use that horizontal slice termination procedure here. Let's see what happens.

D	I	S
$\ln(x)$	x^2	$+$
$\frac{1}{x}$	$\frac{1}{3}x^3$	$-$
		$+$

Figure 15: Integrating $\int x^2 \ln(x)\ dx$: Finally...!

From Figure 15, our integral would be

$$\begin{aligned} I &= \int x^2 \ln(x)\ dx \\ &= \tfrac{1}{3}x^3 \ln(x) - \int \frac{1}{x} \cdot \frac{1}{3}x^3\ dx \end{aligned}$$

Now the point of taking the horizontal slice where we did becomes apparent: We can integrate the horizontal slice in Figure 15! So our integral becomes

$$\begin{aligned} I &= \tfrac{1}{3}x^3 \ln(x) - \int \frac{1}{x} \cdot \frac{1}{3}x^3\ dx \\ &= \tfrac{1}{3}x^3 \ln(x) - \frac{1}{3} \int \frac{x^3}{x}\ dx \\ &= \tfrac{1}{3}x^3 \ln(x) - \tfrac{1}{3} \int x^2\ dx \\ &= \tfrac{1}{3}x^3 \ln(x) - \tfrac{1}{3} \cdot \tfrac{1}{3}x^3 + C \\ &= \tfrac{1}{3}x^3 \ln(x) - \tfrac{1}{9}x^3 + C \\ &= \tfrac{1}{9}x^3 \Big[3\ln(x) - 1\Big] + C \end{aligned}$$

5.3 How to Know Where To Do the Horizontal Slice

We have come across two situations where we've taken a horizontal slice to terminate the infinite series.

The first occurred in Section 5.1 when *the third row in the table was the same as the first row in the table* (apart from the sign). This was significant: the horizontal slice gave us an integral *that was the same as the one we started with*. That enabled us to use algebra to find the value of the integral. So that's a clue to look out for: look for when a row has the same stuff in the D and I columns (apart from signs or coefficients). Another example of this idea can be found in Section 5.5.

The other situation occurred in Section 5.2 where *the second row in the table can be integrated*. That's always going to stop your process in a nice way - if you can integrate a row.

5.4 Why the Horizontal Slice Thing Works

To get an idea of how this idea of integrating a horizontal slice through our table can terminate the infinite process, have a look at Figure 16. Here we are using the DIS method on *the general case*, showing where the IBP formula actually comes from.

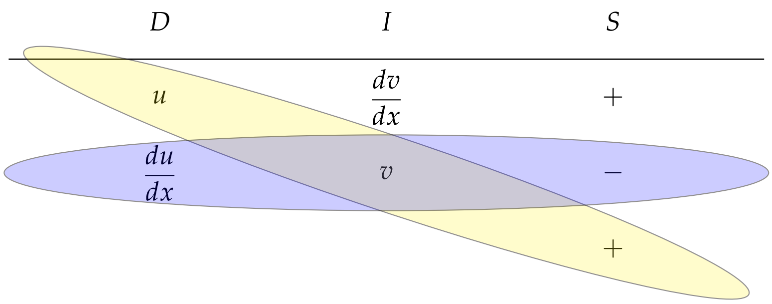

Figure 16: Showing the origin of the IBP formula

Using the DIS method on this example we can see that

$$\int u \frac{dv}{dx}\, dx = uv - \int v \frac{du}{dx}\, dx$$

which is just the IBP formula (1) that we know and love so well.

So the integrating-the-horizontal-slice thing turns out to represent the integral on the right-hand-side of the IBP formula!

And this is also why the signs alternate in the S column. It's because of the $-$ sign in the IBP formula. Each time we go through a cycle of the traditional IBP process, a $-$ sign is introduced into the procedure.

5.5 Two Trigonometrical Functions

The same idea can be used if we have something like

$$I = \int \sin(x)\cos(2x)\, dx$$

to integrate. Again it doesn't really matter which factor we have for D and which for I as they are both periodic functions. See Figure 17 for how to lay out this problem. Again we have to terminate the process after a number of steps by taking a horizontal slice through our table, and integrating it:

$$\begin{aligned} I &= \int \sin(x)\cos(2x)\, dx \\ &= \tfrac{1}{2}\sin(x)\sin(2x) + \tfrac{1}{4}\cos(x)\cos(2x) + \tfrac{1}{4}\int \sin(x)\cos(2x)\, dx \end{aligned}$$

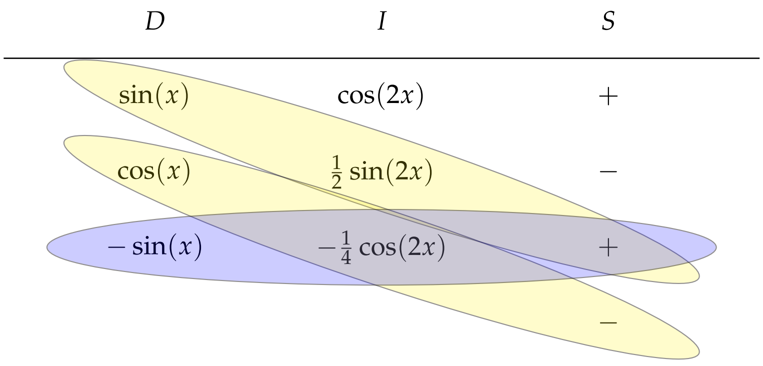

Figure 17: Integrating $\int \sin(x)\cos(2x)\ dx$

Then subtracting $\frac{1}{4}\int \sin(x)\cos(2x)\ dx$ from both sides we get

$$\frac{3}{4}\int \sin(x)\cos(2x)\ dx = \frac{1}{2}\sin(x)\sin(2x) + \frac{1}{4}\cos(x)\ \cos(2x)$$

and so

$$\begin{aligned}\int \sin x\cos 2x\ dx &= \frac{4}{3}\left[\frac{1}{2}\sin(x)\sin(2x) + \frac{1}{4}\cos(x)\cos(2x)\right] + C\\ &= \frac{2}{3}\sin x\sin(2x) + \frac{1}{3}\cos(x)\cos(2x) + C \qquad (4)\end{aligned}$$

Not forgetting the $+C$ of course.

And how did we know when to take the horizontal slice? In row 3 we had the same things ($\sin(x)$ in the D column and $\cos(2x)$ in the I column) as we had to start with (that is, in row 1).

This is a particularly interesting example because the traditional method of solving this kind of integral is to use trigonometrical double / half angle formulae that are rarely used for anything else. Using the DIS method of IBP, you don't have to be so familiar with those identities. Also, because of the nature of trigonometrical identities, the solution obtained using the traditional method would yield

$$\int \sin(x)\cos(2x)\ dx = -\frac{1}{6}\cos(3x) + \frac{1}{2}\cos(x) + C \qquad (5)$$

and if this was a question in an exam, it would take an eagle-eyed examiner to spot that the student-supplied solution (4) was the same as the solution (5) in her mark scheme.

5.6 Powers of Sines and Cosines

It's also possible to use DIS (and any other means of doing integration by parts) to tackle the problem of integrating powers of sines and cosines. Things like

$$I = \int \sin^3(x)\ dx$$

These kinds of integrals are notoriously tricky things to evaluate. Because this kind of thing is a lot more difficult than what we've done so far, I've assigned the details to an Appendix. See Appendix C for more information.

6 Conclusion

DIS is a really cool way of doing integration by parts in a visual way. It's quicker, easier, and prevents a lot of mistakes. And it is *not* taught in schools. Yet[4].

DIS *is* integration by parts, so it *can* be used in A-Level exams. A few years ago, when I first came across this technique, I was worried that an examiner might not give full marks if you used DIS in an exam. So I wrote to the exam boards, all of which (that is, Edexcel, OCR and AQA) assured me that it would be fine to use DIS. If you are a student with an exam looming, and you are still worried about using it, you could always begin your answer with the words "Using the Tabular Integration by Parts method...". That way, if an examiner had never come across the technique, she could simply do an internet search for *Tabular Integration by Parts*, and she would find it. And we would have another convert!

So use it. Wherever you can. And don't be afraid.

Spread the word...

And if you have got this far and want to know what else DIS can do, check out (Smith, 2012). You will be amazed!!

[4]I taught DIS to a student of mine called Zac a few years ago. He immediately went into his school to tell everyone about it, teachers as well as students. DIS is now taught in his school, but's not called DIS. They call it *Zac's Voodoo Method* to this day!

A Standard Forms

Here is a list of standard forms. It's not a comprehensive list, but it includes some important integrals that you either have to know (the top half of the table), or will be in your formula booklet (the bottom half).

Before you have a look at the table, I just wanted to mention a slight bug-bear of mine. It's to do with the name of the abbreviation of the function that's the reciprocal of $\sin(x)$. In the UK, A-Level textbooks use the abbreviation $\text{cosec}(x)$. It's a pretty good abbreviation of the full name of the function, *cosecant*, but it's not very consistent! That would mean that of the of the six circular trigonometrical functions, $\sin(x)$, $\cos(x)$, $\tan(x)$, $\text{cosec}(x)$, $\sec(x)$ and $\cot(x)$, it's the only one that doesn't have a three-letter abbreviation. I know it's a small thing, but I'm a pedant, OK?

In the function bible Abramowitz and Stegun (1972), it is recommended that $\csc(x)$ is used. And this seems to be the abbreviation that's in widespread use throughout the mathematical world. Gellert et al. (1989) is a standard that uses $\text{cosec}(x)$, but it's in the minority. The typesetting language LaTeX (which I use for writing these documents and is the standard for journal articles and PhD theses) only has a code for $\csc(x)$ (which is what reminded me of this whole thing in the first place).

So I'm going to call it $\csc(x)$. Hope that's OK.

Function	**Integral**
ax^n	$\frac{a}{n+1}x^{n+1}$
$\sin(x)$	$-\cos(x)$
$\cos(x)$	$\sin(x)$
e^x	e^x
$\ln(x)$	$\frac{1}{x}$
$\sec^2(kx)$	$\frac{1}{k}\tan(kx)$
$\sec(x)\ \tan(x)$	$\sec(x)$
$\tan(x)$	$\ln(\lvert\sec(x)\rvert)$
$\cot(x)$	$\ln(\lvert\sin(x)\rvert)$
$\csc(x)$	$-\ln(\lvert\csc(x)+\cot(x)\rvert)$
$\sec(x)$	$\ln(\lvert\sec(x)+\tan(x)\rvert)$
$-\csc(x)^2(x)$	$\cot(x)$
$-\csc(x)\ \cot(x)$	$\csc(x)$

Table 1: Integral Standard Forms

B Examples

OK. Here are some examples. These are the most common types of questions that you get on A-Level Maths exams.

B.1 Example 1: $\int 3x \sin\left(x + \frac{\pi}{3}\right)\ dx$

To integrate

$$I = \int 3x \sin\left(x + \frac{\pi}{3}\right)\ dx$$

we use the layout shown in Figure 18.

D	I	S
$3x$	$\sin\left(x + \frac{\pi}{3}\right)$	$+$
3	$-\cos\left(x + \frac{\pi}{3}\right)$	$-$
0	$-\sin\left(x + \frac{\pi}{3}\right)$	$+$
		$-$

Figure 18: Integrating $\int 3x \sin\left(x + \frac{\pi}{3}\right)\ dx$

This gives

$$\begin{aligned} I &= \int 3x \sin\left(x + \frac{\pi}{3}\right)\ dx \\ &= -3x \cos\left(x + \frac{\pi}{3}\right) + 3 \sin\left(x + \frac{\pi}{3}\right) + C \end{aligned}$$

B.2 Example 2: $\int 2xe^{3x+1}\,dx$

To integrate

$$I = \int 2xe^{3x+1}\,dx$$

we use the layout shown in Figure 19.

D	I	S
$2x$	e^{3x+1}	$+$
2	$\frac{1}{3}e^{3x+1}$	$-$
0	$\frac{1}{9}e^{3x+1}$	$+$
		$-$

Figure 19: Integrating $\int 2xe^{3x+1}\,dx$

This gives

$$\begin{aligned} I &= \int 2xe^{3x+1}\,dx \\ &= \tfrac{2}{3}xe^{3x+1} - \tfrac{2}{9}e^{3x+1} + C \\ &= \tfrac{2}{9}e^{3x+1}\Big[3x - 1\Big] + C \end{aligned}$$

B.3 Example 3: $\int \frac{x}{2e^x}\,dx$

To integrate

$$I = \int \frac{x}{2e^x}\,dx$$

we use the layout shown in Figure 20, noticing that

$$\frac{1}{2e^x} = \frac{1}{2}\frac{1}{e^x} = \frac{1}{2}e^{-x}$$

D	I	S
x	$\frac{1}{2}e^{-x}$	$+$
1	$-\frac{1}{2}e^{-x}$	$-$
0	$\frac{1}{2}e^{-x}$	$+$
		$-$

Figure 20: Integrating $\int \frac{x}{2e^x}\,dx$

This gives

$$\begin{aligned} I &= \int \frac{x}{2e^x}\,dx \\ &= -\tfrac{1}{2}xe^{-x} - \tfrac{1}{2}e^{-x} + C \\ &= -\tfrac{1}{2}e^{-x}\left[x + 1\right] + C \end{aligned}$$

B.4 Example 4: $\int x \csc^2(x)\ dx$

See Appendix A for why I've used the abbreviation $\csc(x)$.

To integrate

$$I = \int x \csc^2(x)\ dx$$

we use the layout shown in Figure 21.

D	I	S
x	$\csc^2(x)$	$+$
1	$-\cot(x)$	$-$
0	$-\ln\left[\sin(x)\right]$	$+$
		$-$

Figure 21: Integrating $\int x \csc^2(x)\ dx$

This gives

$$\begin{aligned} I &= \int x \csc^2(x)\ dx \\ &= -x \cot(x) + \ln\Big[\sin(x)\Big] + C \end{aligned}$$

How did I know that the integral of $\csc^2(x)$ was $-\cot(x)$, and the integral of $\cot(x)$ was $\ln\left[\sin(x)\right]$? They're in the formula book, so they are *standard forms*.

B.5 Example 5: $\int x \ln(x)\, dx$

To integrate

$$I = \int x \ln(x)\, dx$$

we use the layout shown in Figure 22.

D	I	S
$\ln(x)$	x	$+$
$\frac{1}{x}$	$\frac{1}{2}x^2$	$-$
		$+$

Figure 22: Integrating $\int x \ln(x)\, dx$

This is an interesting one. We can't integrate $\ln(x)$ directly[5]. So we must differentiate the $\ln(x)$, and integrate the x. This gives

$$\begin{aligned} I &= \int x \ln(x)\, dx \\ &= \tfrac{1}{2}x^2 \ln(x) - \int \frac{1}{x}\frac{1}{2}x^2\, dx \\ &= \tfrac{1}{2}x^2 \ln(x) - \int \tfrac{1}{2}x\, dx \\ &= \tfrac{1}{2}x^2 \ln(x) - \tfrac{1}{4}x^2 + C \\ &= \tfrac{1}{4}x^2 \Big[2\ln(x) - 1\Big] + C \end{aligned}$$

And how did we know when to take the horizontal slice? We can integrate row 2!

[5] Actually, the way to integrate $\ln(x)$ is to use IBP: put $\ln(x)$ in the D column and 1 in the I column! See Example B.7.

B.6 Example 6: $\int e^{4x} \sin(3x)\ dx$

This example is on the extremely hard side as far as A-Level Maths questions go. It's very unlikely to come up. But you never know...

To integrate

$$I = \int e^{4x} \sin(3x)\ dx$$

we use the layout shown in Figure 23.

D	I	S
e^{4x}	$\sin(3x)$	$+$
$4e^{4x}$	$-\frac{1}{3}\cos(3x)$	$-$
$16e^{4x}$	$-\frac{1}{9}\sin(3x)$	$+$
		$-$

Figure 23: Integrating $\int e^{4x} \sin(3x)\ dx$

This gives

$$\int e^{4x} \sin(3x)\ dx = -\tfrac{1}{3}e^{4x}\cos(3x) + \tfrac{4}{9}e^{4x}\sin(3x) - \int \tfrac{16}{9}e^{4x}\sin(3x)\ dx$$

So we can add $\int \frac{16}{9}e^{4x}\sin(3x)\ dx$ to both sides:

$$\begin{aligned}
\tfrac{25}{9}\int e^{4x} \sin(3x)\ dx &= -\tfrac{1}{3}e^{4x}\cos(3x) + \tfrac{4}{9}e^{4x}\sin(3x) \\
\Rightarrow \int e^{4x} \sin(3x)\ dx &= \tfrac{9}{25}\left[-\tfrac{1}{3}e^{4x}\cos(3x) + \tfrac{4}{9}e^{4x}\sin(3x)\right] \\
&= -\tfrac{3}{25}e^{4x}\cos(3x) + \tfrac{4}{25}e^{4x}\sin(3x) + C \\
&= \frac{e^{4x}}{25}\Big[4\sin(3x) - 3\cos(3x)\Big] + C
\end{aligned}$$

And how did we know when to take the horizontal slice? In row 3 there are the same things (e^{4x} in the D, $\sin(3x)$ in the I) in the D and I columns as in row 1 (apart from the signs and coefficients).

B.7 Example 7: $\int \ln(x)\ dx$

The first problem when you want to integrate

$$I = \int \ln(x)\ dx$$

is that *there is no product*! So what do we do? Well, there's a neat trick here: we can *make* a product by multiplying $\ln(x)$ by 1!

So we can use the layout shown in Figure 24.

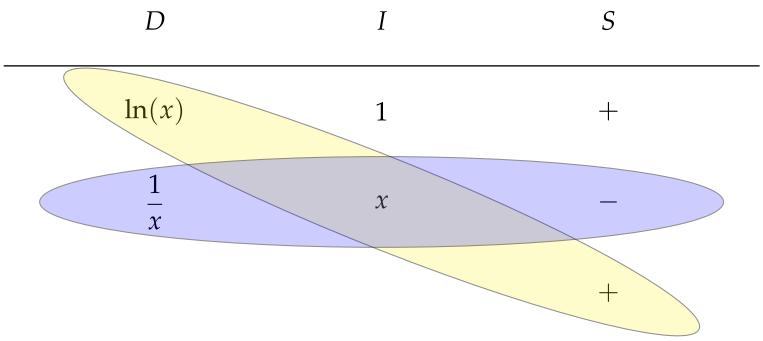

Figure 24: Integrating $\int \ln(x)\ dx$

Again, we can't integrate $\ln(x)$ directly. So we must differentiate the $\ln(x)$, use a 1 to provide the product, and integrate the 1. This gives

$$\begin{aligned} I &= \int \ln(x)\ dx \\ &= x\ln(x) - \int \frac{1}{x} x\ dx \\ &= x\ln(x) - \int 1\ dx \\ &= x\ln(x) - x + C \\ &= x\Big[\ln(x) - 1\Big] + C \end{aligned}$$

And how did we know when to take the horizontal slice? We can integrate row 2!

This idea of integrating something by knowing how to differentiate it is quite a cunning one, isn't it!

B.8 Example 8

Figure 25 shows the curve $y = x\cos(x)$ for x between -2π and 2π. Find the areas A, B and C.

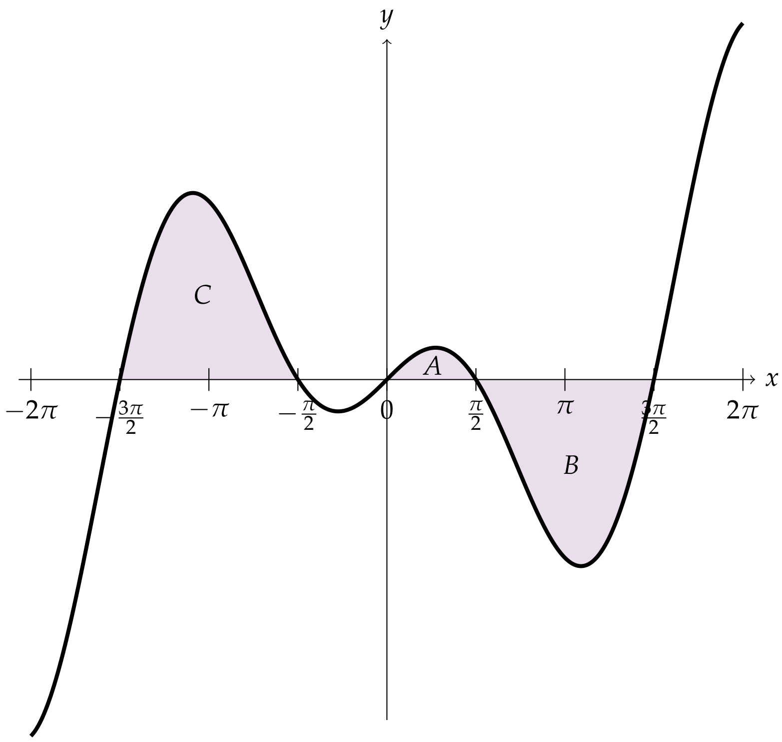

Figure 25: $y = x\cos(x)$

Well, in order to find areas under curves we need to integrate the function. So we will need to evaluate

$$I = \int x\cos(x)\ dx$$

for each area. This can be done using DIS, following the scheme outlined in Figure 26.

D	I	S
x	$\cos(x)$	$+$
1	$\sin(x)$	$-$
0	$-\cos(x)$	$+$
		$-$

Figure 26: Integrating $\int x\cos(x)\ dx$

This gives

$$\int x\cos(x)\ dx = x\sin(x) + \cos(x)$$

Now in order to work out a definite area, we need limits on our integral. Area A has limits of 0 and $\frac{\pi}{2}$, so

$$\begin{aligned}
A &= \int_{x=0}^{x=\frac{\pi}{2}} x\cos(x)\ dx \\
&= \Big[x\sin(x)+\cos(x)\Big]_{x=0}^{x=\frac{\pi}{2}} \\
&= \left[\frac{\pi}{2}\sin\left(\frac{\pi}{2}\right)+\cos\left(\frac{\pi}{2}\right)\right]-\Big[0\sin(0)+\cos(0)\Big] \\
&= \left[\frac{\pi}{2}+0\right]-\Big[0+1\Big] \\
&= \frac{\pi}{2}-1
\end{aligned}$$

Similarly,

$$\begin{aligned}
B &= \int_{x=\frac{\pi}{2}}^{x=\frac{3\pi}{2}} x\cos(x)\ dx \\
&= \Big[x\sin(x)+\cos(x)\Big]_{x=\frac{\pi}{2}}^{x=\frac{3\pi}{2}} \\
&= \left[\frac{3\pi}{2}\sin\left(\frac{3\pi}{2}\right)+\cos\left(\frac{3\pi}{2}\right)\right]-\left[\frac{\pi}{2}\sin\left(\frac{\pi}{2}\right)+\cos\left(\frac{\pi}{2}\right)\right] \\
&= \left[-\frac{3\pi}{2}+0\right]-\left[\frac{\pi}{2}+0\right] \\
&= -\frac{3\pi}{2}-\frac{\pi}{2} \\
&= -2\pi
\end{aligned}$$

Now the reason that this integral has turned out to be negative is that the area is wholly *below* the x-axis. That's just a quirk of the integration process. The area will thus just be 2π.

And because this function $x\cos(x)$ is the product of an odd function and an even function, it is odd itself, and hence rotationally symmetric. That means that area C will be the same as area B.

C Powers of Sines and Cosines

IBP isn't the standard method to evaluate these integrals at A-Level. But it shows the power of IBP that you can use it to do even these.

C.1 The Big Idea

Let's leap straight in here with the idea of trying to integrate a general power of $\sin(x)$:

$$I_n = \int \sin^n(x) \; dx$$

First of all, take a look at that notation. Here, I_n represents the integral of the n^{th} power of $\sin(x)$. Using the same idea, I_{n-2} would represent the integral of the $(n-2)^{th}$ power of $\sin(x)$, etc, etc.

In order to do this integral, I lay out my table as shown in Figure 27.

D	I	S
$\sin^{n-1}(x)$	$\sin(x)$	$+$
$(n-1)\sin^{n-2}(x)\cos(x)$	$-\cos(x)$	$-$
		$+$

Figure 27: Integrating $\int \sin^n(x) \; dx$

Notice the cunning way that I've developed a product:

$$\int \sin^n(x)\, dx = \int \sin^{n-1}(x)\sin(x)\, dx$$

Sneaky, eh? Again we have to terminate the process after a number of steps (this time, only one step!) by taking a horizontal slice through our table, and integrating it:

$$\begin{aligned} I_n &= \int \sin^n(x)\, dx \\ &= -\sin^{n-1}(x)\cos(x) + \int (n-1)\sin^{n-2}(x)\cos^2(x)\, dx \\ &= -\sin^{n-1}(x)\cos(x) + \int (n-1)\sin^{n-2}(x)\left[1-\sin^2(x)\right] \; dx \end{aligned}$$

because $\sin^2(x) + \cos^2(x) = 1$. So,

$$\begin{aligned} I_n &= -\sin^{n-1}(x)\cos(x) + \int (n-1)\sin^{n-2}(x)\, dx - \int (n-1)\sin^{n-2}(x)\sin^2(x)\, dx \\ &= -\sin^{n-1}(x)\cos(x) + (n-1)\int \sin^{n-2}(x)\, dx - (n-1)\int \sin^n(x)\, dx \\ &= -\sin^{n-1}(x)\cos(x) + (n-1)I_{n-2} - (n-1)I_n \end{aligned}$$

using that notation idea, and so by adding $(n-1)I_n$ to both sides we get

$$nI_n = -\sin^{n-1}(x)\cos(x) + (n-1)I_{n-2}$$

which, by dividing both sides by n, leads to a recurrence relation

$$I_n = -\left[\frac{1}{n}\right] \sin^{n-1}(x)\cos(x) + \left[\frac{n-1}{n}\right] I_{n-2} \tag{6}$$

that you can use to find the integral of any power of $\sin(x)$. The same sort of thing can be done for powers of cosines, too. For powers of cosines, the recurrence relation turns out to be

$$I_n = \left[\frac{1}{n}\right] \cos^{n-1}(x)\sin(x) + \left[\frac{n-1}{n}\right] I_{n-2} \tag{7}$$

which is pretty similar.

C.2 A Simple(!) Example

To see how we could use this recurrence relation, lets try integrating

$$I_3 = \int \sin^3(x) \;\; dx$$

OK, so using our recurrence relation (6), we would get

$$I_3 = -\tfrac{1}{3}\sin^2(x)\cos(x) + \tfrac{2}{3} I_1$$

But I_1 is just the integral of $\sin(x)$, so this becomes

$$\begin{aligned} I_3 &= -\tfrac{1}{3}\sin^2(x)\cos(x) - \tfrac{2}{3}\cos(x) + C \\ &= -\tfrac{1}{3}\cos(x)\left[\sin^2(x) + 2\right] + C \end{aligned} \tag{8}$$

Again, the customary A-Level approach to solving this integral leads to the solution

$$I_3 = -\tfrac{1}{12}\left[\cos(3x) - 9\cos(x)\right] + C \tag{9}$$

and it's pretty hard to spot that equations (8) and (9) are indeed the same.

So you can use DIS to provide an entry into the ideas of using recurrence relations to solve integrals. This used to be on the A-Level maths syllabus, but seems to have fallen by the wayside. Shame.

C.3 A More Difficult Example

This time, lets try integrating

$$I_4 = \int \sin^4(x) \;\; dx$$

OK, so using our recurrence relation (6), we would get

$$I_4 = -\tfrac{1}{4}\sin^3(x)\cos(x) + \tfrac{3}{4} I_2$$

But we can use the recurrence relation again to find I_2:

$$I_4 = -\tfrac{1}{4}\sin^3(x)\cos(x) + \tfrac{3}{4}\left[-\tfrac{1}{2}\sin(x)\cos(x) + \tfrac{1}{2} I_0\right]$$

But I_0 is just the integral of 1 ($I_0 = \int sin^0(x)dx = \int 1dx$), so

$$\begin{aligned} I_4 &= -\tfrac{1}{4}\sin^3(x)\cos(x) + \tfrac{3}{4}\left[-\tfrac{1}{2}\sin(x)\cos(x) + \tfrac{1}{2}x\right] + C \\ &= -\tfrac{1}{4}\sin^3(x)\cos(x) - \tfrac{3}{8}\sin(x)\cos(x) + \tfrac{3}{8}x + C \end{aligned}$$

And actually, that wasn't so bad. I hope that you can see how this sort of thing works.

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